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Bourgin-Yang versions of the Borsuk-Ulam theorem for (H, G)-coincidences

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Let $G = \mathbb{Z}_{p^k}$ be a cyclic group of prime power order, or repectively $G = \mathbb{Z}_p^k$ be the p-torus of rank k. We estimate the size of the (H, G)-coincidences set of a continuous map from $S(V)$ into a real vector space W'. May, 2016 ICMC-USP

1. INTRODUCTION

Let G be a finite group which acts freely on a space X and let $f : X \to Y$ be a continuous map from X into another space Y. If H is a subgroup of G, then H acts on the right on each orbit Gx of G as follows: if $y \in Gx$ and $y = gx$, $g \in G$, then $hy = gh^{-1}x$. A point $x \in X$ is said to be a (H, G)- *coincidence point* of f (as introduced by Gonçalves, Pergher and Jaworowski in [3]) if f sends every orbit of the action of H on the G-orbit of x to a single point. Let us denote by $A(f, H, G)$ the set of all (H, G)-coincidence points. Of course, if H is the trivial subgroup, then every point of X is a (H, G)-coincidence. If $H = G$, this is the usual definition of G-coincidence, that is,

$$A(f, G, H) = A(f) = \{x \in X \mid f(x) = f(gx), \text{ for all } g \in G\}.$$

Borsuk-Ulam theorems type consists in estimating the dimension of the set $A(f, H, G)$. For the case that $G = H = \mathbb{Z}_2$, $X = S^n$ and $Y = \mathbb{R}^n$, we have the classical Borsuk-Ulam theorem [2].

Let $G = \mathbb{Z}_{p^k}$ be a cyclic group of prime power order, $k \geq 1$. For given two powers $1 \leq m \leq n \leq p^{k-1}$ of p we set

$$\mathcal{A}_{m,n} := \{G/H \mid H \subset G;\ m \leq |H| \leq n\}, \tag{1}$$

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where $|H|$ is the cardinality of H. We shall write $\mathcal{A}_X$ for a set of all the G-orbits of a space X (up to a homeomorphism, thus up to an isomorphism of finite G-sets).

Let V be an orthogonal representation of $G = \mathbb{Z}_{p^k}$, p prime, $k \geq 1$, such that $V^G = \{0\}$, for the set of fixed points of G. For $G = \mathbb{Z}_{p^k}$, with p odd, every nontrivial irreducible orthogonal representation is even dimensional and admits the complex structure ([4]), thus V admit it too. We denote by $d(V) = \dim_{\mathbb{C}} V = \frac{1}{2}\dim_{\mathbb{R}} V$ the integral numerical invariant of V. If $G = \mathbb{Z}_{2^k}$ and V is a real orthogonal representation of G, then we denote $d(V) = \dim_{\mathbb{R}} V$.

Given W' a real vector space and a continuous map $f : S(V) \to W'$, in this work we estimate the size of $A(f, \mathbb{Z}_{p^i}, \mathbb{Z}_{p^k})$ the $\mathbb{Z}_{p^k}$-coincidences set of f, as follows.

THEOREM 1.1.1. *Let V be a complex orthogonal representation of the cyclic group $G = \mathbb{Z}_{p^k}$, p prime, $k \geq 1$, such that $V^G = \{0\}$ and let W' be a real vector space. Let $f : S(V) \to W'$ be a continuous map.*

(1)If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,p^{k-1}}$, then for all $1 \leq i \leq k$,

$$\dim A(f, \mathbb{Z}_{p^i}, \mathbb{Z}_{p^k}) \geq 2\left(\frac{d(V)-1}{p^{k-1}}\right) - (p^k - p^{k-i})d(W').$$

(2)If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,p^{i-1}}$ for some $1 \leq i \leq k$, then

$$\dim A(f, \mathbb{Z}_{p^i}, \mathbb{Z}_{p^k}) \geq 2\left(\frac{d(V)-1}{p^{i-1}}\right) - (p^k - p^{k-i})d(W').$$

THEOREM 1.1.2. *Let V be a real orthogonal representation of the cyclic group $G = \mathbb{Z}_{2^k}$, $k \geq 1$, such that $V^G = \{0\}$ and let W' be a real vector space. Let $f : S(V) \to W'$ be a continuous map.*

(1)If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,2^{k-1}}$, then for all $1 \leq i \leq k$,

$$\dim A(f, \mathbb{Z}_{2^i}, \mathbb{Z}_{2^k}) \geq \left(\frac{d(V)-1}{2^{k-1}}\right) - (2^k - 2^{k-i})d(W').$$

(2)If $\mathcal{A}_{S(V)} \subset \mathcal{A}_{1,2^{i-1}}$ for some $1 \leq i \leq k$, then

$$\dim A(f, \mathbb{Z}_{2^i}, \mathbb{Z}_{2^k}) \geq \left(\frac{d(V)-1}{2^{i-1}}\right) - (2^k - 2^{k-i})d(W').$$

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2. BOURGIN-YANG VERSIONS OF THE BORSUK-ULAM THEOREM FOR $G = \mathbb{Z}_{p^K}$

Recently, in [5], the authors proved the following Bourgin-Yang version of the Borsuk-Ulam theorem for complex orthogonal representations of $G = \mathbb{Z}_{p^k}$, p prime, $k \geq 1$

THEOREM 2.2.1. *[5, Theorem 3.6] Let V, W be two complex orthogonal representations of the cyclic group $G = \mathbb{Z}_{p^k}$, p prime, $k \geq 1$, such that $V^G = W^G = \{0\}$. Let $f : S(V) \overset{G}{\to} W$ be an equivariant map and $Z_f := f^{-1}(0) = \{v \in S(V) \,|\, f(v) = 0\}$. Suppose $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ and $\mathcal{A}_{S(W)} \subset \mathcal{A}_{m,n}$. Then*

$$\dim(Z_f) \;\geq\; 2\Big(\Big(\frac{(d(V)-1)m}{n}\Big) - d(W)\Big).$$

They also proved the following Bourgin–Yang version of the Borsuk-Ulam theorem for real orthogonal representations of $G = \mathbb{Z}_{2^k}$, $k \geq 1$.

THEOREM 2.2.2. *[5, Theorem 3.9] Let V, W be two real orthogonal representations of the cyclic group $G = \mathbb{Z}_{2^k}$, $k \geq 1$, such that $V^G = W^G = \{0\}$. Let $f : S(V) \overset{G}{\to} W$ be an equivariant map and $Z_f = f^{-1}(0)$. Suppose that $\mathcal{A}_{S(V)} \subset \mathcal{A}_{m,n}$ and $\mathcal{A}_{S(W)} \subset \mathcal{A}_{m,n}$. Then*

$$\dim(Z_f) \;\geq\; \Big(\frac{(d(V)-1)m}{n}\Big) - d(W).$$

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1.1 Let i be fixed, with $1 \leq i \leq k$ and let us consider the real vector space $\overset{p^k}{\underset{j=1}{\oplus}} W'$, which is the direct sum of p^k copies of W'. We have that $\overset{p^k}{\underset{j=1}{\oplus}} W'$ admits an action of the cyclic group $G = \mathbb{Z}_{p^k}$, given by

$$g(w_1, w_2, \ldots, w_{p^k}) = (w_2, \ldots, w_{p^k}, w_1),$$

for a fixed generator $g \in G$ and for each $(w_1, \ldots, w_{p^k}) \in \overset{p^k}{\underset{j=1}{\oplus}} W'$.

Let us denote by $\Delta(W'^{p^{k-i}})$ the diagonal of

$$\overset{p^k}{\underset{j=1}{\oplus}} W' = W'^{p^{k-i}} \oplus \cdots \oplus W'^{p^{k-i}}.$$

We have

$$\overset{p^k}{\underset{j=1}{\oplus}} W' = \Delta(W'^{p^{k-i}}) \oplus (\Delta(W'^{p^{k-i}}))^{\perp},$$

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where $\Delta(W'^{p^{k-i}})^{\perp}$ is the orthogonal complement of $\Delta(W'^{p^{k-i}})$. Since $\Delta(W'^{p^{k-i}})$ is a $p^{k-i}\dim W'$ - dimensional G-subspace of $\overset{p^k}{\underset{i=1}{\oplus}} W'$, let us observe that $\Delta(W^{p^{k-i}})^{\perp}$ is a $(p^k - p^{k-i})\dim W'$ - dimensional G-subrepresentation of $\overset{p^k}{\underset{i=1}{\oplus}} W'$, for which $(\Delta(W'^{p^{k-i}})^{\perp})^G = \{0\}$.

Now, we denote by $a_1, \ldots, a_r$ a set of representatives of the left lateral classes of $G/\mathbb{Z}_{p^i}$, where $r = p^{k-i}$. Consider the map

$$F : S(V) \to \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp}$$

defined by

$$F(x) = (f(a_1x), \ldots, f(a_rx), f(a_1hx), \ldots, f(a_rhx), \ldots, f(a_1h^{p^i-1}x), \ldots, f(a_rh^{p^i-1}x)),$$

for a fixed generator $h \in \mathbb{Z}_{p^i}$. The linear orthogonal projection along the diagonal $\Delta(W'^{p^{k-i}})$ defines a G-equivariant map $\rho : \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp} \to \Delta(W'^{p^{k-i}})^{\perp}$. Let us denote by l the composition

$$S(V) \xrightarrow{F} \Delta(W'^{p^{k-i}}) \oplus \Delta(W'^{p^{k-i}})^{\perp} \xrightarrow{\rho} \Delta(W'^{p^{k-i}})^{\perp},$$

with $Z_l = l^{-1}(0) = (\rho \circ F)^{-1}(0) = F^{-1}(\Delta(W'^{p^{k-i}})) = A(f, \mathbb{Z}_{p^i}, \mathbb{Z}_{p^k})$.

For a fixed generator $g \in G$, we can consider

$$h = g^{p^{k-i}} \quad \text{and} \quad a_1 = e, a_2 = g, \ldots, a_r = g^{p^{k-i}-1},$$

then F is a G-equivariant map. Moreover

$$\mathcal{A}_{S(\Delta(W'^{p^{k-i}})^{\perp})} \subset \mathcal{A}_{1,p^{i-1}} \subset \mathcal{A}_{1,p^{k-1}}. \tag{1}$$

To check the validity of inclusion $\mathcal{A}_{S(\Delta(W'^{p^{k-i}})^{\perp})} \subset \mathcal{A}_{1,p^{i-1}}$, it suffices to prove that the cardinality of the orbit $\mathbb{Z}_{p^k}w$ belongs to the set $\{p^k, p^{k-1}, \ldots, p^{k-i+1}\}$, for all $w = (w_1, \ldots, w_{p^k}) \in S(\Delta(W'^{p^{k-i}})^{\perp})$. According to [1, Chapter 1, Proposition 4.1], the cardinality of the orbit $\mathbb{Z}_{p^k}w$ belongs to the set $\{p^k, p^{k-1}, \ldots, p, p^0 = 1\}$. Let $w = (w_1, \ldots, w_{p^k})$ an element in $S(\Delta(W'^{p^{k-i}})^{\perp})$ and suppose that

$$|\mathbb{Z}_{p^k}w| \in \{p^{k-i}, p^{k-i-1}, \ldots, p^0 = 1\},$$

that is $|\mathbb{Z}_{p^k}w| = p^j$, for some $0 \le j \le k - i$.

Assertion. We have $\mathbb{Z}_{p^k}w = \{w, gw, \cdots, g^{p^j-1}w\}$, for a fixed generator g of $\mathbb{Z}_{p^k}$.

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In fact, consider a cyclic group G, $g \in G$ a fixed generator and $\{w, gw, \cdots, g^{s-1}w\}$ the maximum set of the s first elements of the orbit Gw that are distinct from each other. By definition of the set $\{w, gw, \cdots, g^{s-1}w\}$ we have

$$g^s w \in \{w, gw, \cdots, g^{s-1}w\}.$$

Suppose that

$$g^s w = g^i w, \text{ for some } 1 \leq i \leq s-1,$$

then

$$g^{s-i} w = w, \ 1 \leq s-i \leq s-1,$$

but this contradicts the definition of $\{w, gw, \cdots, g^{s-1}w\}$.

Now, if $g^t w \in Gw$, for some $t \in \mathbb{N}$, we have $t = ns + r$ with $0 \leq r \leq s-1$. Therefore

$$g^t w = g^{ns+r} w = g^r(g^{ns})w = g^r w \in \{w, gw, \cdots, g^{s-1}w\},$$

since

$$g^{ns} w = (g^s \cdots g^s)w = w$$

and $0 \leq r \leq s-1$.

Thus, for a fixed generator g of $\mathbb{Z}_{p^k}$, we have

$$\begin{aligned} w = g^{p^j} w &= g^{p^j}(w_1, \ldots, w_{p^j}, \ldots, w_{(p^{k-j}-1)p^j+1}, \ldots, w_{p^k}) \\ &= (w_{p^j+1}, \ldots, w_{2p^j}, \ldots, w_{(p^{k-j}-1)p^j+1}, \ldots, w_{p^k}, w_1, \ldots, w_{p^j}) \end{aligned}$$

therefore $w \in \Delta(W'^{p^j})$. Since

$$\Delta(W') \subset \Delta(W'^p) \subset \cdots \subset \Delta(W'^{p^{k-i-1}}) \subset \Delta(W'^{p^{k-i}})$$

and $j \in \{0, 1, \ldots, k-i\}$, we conclude that

$$w \in \Delta(W'^{p^j}) \subset \Delta(W'^{p^{k-i}}),$$

which is a contradiction because

$$\Delta(W'^{p^{k-i}}) \cap S(\Delta(W'^{p^{k-i}})^\perp) = \emptyset.$$

Thus the Theorem 2.2.1 implies the claim. □

Proof of Theorem 1.1.2 For $G = \mathbb{Z}_{2^k}$, $k \geq 1$, using the same steps of the proof of Theorem 1.1.1 and applying Theorem 2.2.2 we have the result. □

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REMARK 3.3.1. *We emphasize that in the Theorems 1.1.1 and 1.1.2 the action of G on $S(V)$ is not necessarily free. Moreover, we have an estimate for the size of the set of (H,G)-coincidence points for all subgroups $H = \mathbb{Z}_{p^i}$ of $G = \mathbb{Z}_{p^i}$. These two characteristics make the Theorems 1.1.1 and 1.1.2 different of classical results about (H,G)-coincidences.*

4. CASE OF G BEING A P-TORUS

Now, let V, W be two orthogonal representations of the p-torus group $G = \mathbb{Z}_p^k$ of rank $k \geq 1$, p prime, such that $V^G = W^G = \{0\}$ for the sets of fixed points of G. Let $f : S(V) \to W'$ be a G-equivariant map, in [6, Theorem 2.7] we have the following estimate for the covering dimension of Z_f

$$\dim(Z_f) \geq d(V) - d(W) - 1.$$

Using this result we can estimate the covering dimension of the set of $(H, \mathbb{Z}_p^k)$-coincidences, as follows.

COROLLARY 4.4.1. *Let V, W' be two orthogonal representation of the group $G = \mathbb{Z}_p^k$, such that $V^G = {W'}^G = \{0\}$. Let $f : S(V) \to W'$ be a G-equivariant map, then*

$$\dim A(\mathbb{Z}_p^i, \mathbb{Z}_p^k) \geq d(V) - (p^k - p^{k-i})d(W') - 1$$

for all $1 \leq i \leq k$.

Proof: Let i be fixed, with $1 \leq i \leq k$. As in the proof of the Theorem 1.1.1, we consider a map

$$F : S(V) \to \bigoplus_{j=1}^{p^k} W'$$

defined by

$$F(x) = (f(a_1h_1x), \ldots, f(a_rh_1x), f(a_1h_2x), \ldots, f(a_rh_2x), \ldots, f(a_1h_{p^i}x), \ldots, f(a_rh_{p^i}x)),$$

where $a_1, \ldots, a_r$ are representatives of the left lateral classes of $G/\mathbb{Z}_p^i$, $r = p^{k-i}$ and $\mathbb{Z}_p^i = \{h_1, \ldots, h_{p^i}\}$.

Considering the diagonal action of G on $\bigoplus_{j=1}^{p^k} W'$, F becomes G-equivariant. Thus the composition

$$S(V) \xrightarrow{F} \Delta({W'}^{p^{k-i}}) \oplus \Delta({W'}^{p^{k-i}})^{\perp} \xrightarrow{\rho} \Delta({W'}^{p^{k-i}})^{\perp},$$

is also G-equivariant. Finally, we apply [6, Theorem 2.7] to estimate the size of the set $Z_{\rho \circ F} = A(\mathbb{Z}_p^i, \mathbb{Z}_p^k)$. □

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REFERENCES

1. Bredon, G., *Introduction to compact transformation groups*, Vol. **46**, Academic Press, New York and London, (1972).
2. Borsuk, K., *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20** 177–190, 1933.
3. Gonçalves, D.L.; Jaworowski, J.; Pergher, Pedro L.Q., *Measuring the size of the coincidence set*, Topology Appl. **125** (2002), 465–470.
4. Serre, J. P. *Linear representations of finite groups*. Translated from the second French edition by Leonard L. Scott., Graduate Texts in Mathematics 42, Springer-Verlag, New York-Heidelberg, (1977).
5. Waclaw Marzantowicz, D. de Mattos, Edivaldo L. dos Santos, *Bourgin-Yang version of the Borsuk-Ulam theorem for* $\mathbb{Z}_{p^k}$*-equivariant maps*, Algebraic & Geometric Topology, to appear 2012.
6. Waclaw Marzantowicz, D. de Mattos, Edivaldo L. dos Santos, *Bourgin-Yang versions of the Borsuk-Ulam theorem for p-toral groups.*

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