Positive definite kernels on complex spheres

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Continuous bizonal positive definite kernels on the spheres in \( \mathbb{C}^q \) are shown to be series of disk polynomials with nonnegative coefficients. These kernels are the complex analogs of the so-called positive definite functions on real spheres introduced and characterized by I. J. Schoenberg [Duke Math. J. 9 (1942), 96-108]. The result adds to the classes of functions from which one can generate radial basis function interpolants to arbitrary data on spheres. April, 2000 ICMC-USP

1. INTRODUCTION

We are interested in positive definite kernels on the unit sphere \( \Omega_{2q} \) in \( \mathbb{C}^q \) with emphasis on those continuous ones having the form

\[(\xi, \zeta) \in \Omega_{2q}^2 \rightarrow f((\xi, \zeta)).\]

In this expression, \((\cdot, \cdot)\) denotes the usual inner product in \( \mathbb{C}^q \) and \( f \) is a complex function defined on \( \Omega_2 \) when \( q = 1 \) and on the closed unit disk \( B_2 \) in \( \mathbb{C} \) otherwise. Kernels of the above form are called bizonal due to the fact that they become zonal functions on \( \Omega_{2q} \) when one of the variables is fixed. A function is \( \zeta \)-zonal if it is of the form \( \xi \in \Omega_{2q} \rightarrow g((\xi, \zeta)) \) for some function \( g \) defined on \( B_2 \) (\( \Omega_2 \), if \( q = 1 \)). This concept agrees with that appearing in many books dealing with analysis on real spheres ([7,10,13]).

We recall that a kernel \( K: A^2 \rightarrow \mathbb{C} \) is positive definite on a nonempty set \( A \) if and only if

\[ \sum_{\mu, \nu=1}^{t} c_{\mu} c_{\nu} K((\xi_\mu, \xi_\nu) \geq 0, \]

where \( c_{\mu} c_{\nu} \) are complex coefficients.
for all $l \in \mathbb{N} := \{0, 1, \ldots \}$, $\{\xi_1, \xi_2, \ldots, \xi_l\} \subset A$ and $\{c_1, c_2, \ldots, c_l\} \subset \mathbb{C}$. In other words, $K$ is positive definite on $A$ if and only if every matrix of the form $(K(\xi_\mu, \xi_\nu))$, $\{\xi_1, \xi_2, \ldots, \xi_l\} \subset A$, is nonnegative definite. It is convenient to call a function $f$ positive definite on $A \subset \Omega_{2q}$ whenever the corresponding bizonal kernel $f((\cdot, \cdot))$ is positive definite on $A$.

Our primary interest in bizonal kernels is due to the possibility of using them to perform interpolation on $\Omega_{2q}$. Under reasonable additional hypotheses on the positive definite function $f$ on $\Omega_{2q}$, it is possible to guarantee the positive definiteness of many matrices of the form $(f((\xi_\mu, \xi_\nu)))$ and as a consequence interpolation of arbitrary data at the $\xi_\mu$ by a function of the form

$$
\xi \in \Omega_{2q} \rightarrow \sum_{\mu=1}^{l} c_\mu f((\xi, \xi_\mu)),
$$

can be performed. When the points $\xi_\mu$ belong to a real sphere contained in $\Omega_{2q}$ then the restriction of the above interpolant to that sphere becomes a spherical radial basis function or spherical spline; the reader is referred to [3,6] and references therein for acquaintance with this recent and important field of research in approximation theory.

The main purpose of this paper is to present a complete characterization of continuous bizonal positive definite kernels on $\Omega_{2q}$ thus generalizing a famous result of I. J. Schoenberg ([11]) on positive definite functions on real spheres. In addition to that, the result expands the class of functions that can be used to construct spherical radial basis function interpolants to arbitrary data on real spheres. Our approach has similarities with that of Schoenberg but the arguments here are more complicated. In the present situation, we will deal with a class of homogeneous, zonal and harmonic polynomials in $\mathbb{C}^q$ and with the Poisson-Szegő kernel for the unit ball $B_{2q}$ of $\mathbb{C}^q$. The above mentioned polynomials are connected to the so-called disk polynomials which are two dimensional analogs of Jacobi polynomials ([1,9]) while the Poisson-Szegő kernel generates solutions to certain Dirichlet problems on $B_{2q}$ and is connected to Jacobi polynomials and certain hypergeometric functions ([12]).

The paper is organized as follows. In Section 2, we introduce notation and present necessary technical lemmata. In Section 3, we characterize positive definite bizonal kernels on $\Omega_{2q}$. In Section 4, taking Section 3 as a model, we characterize the positive definite bizonal kernels on $\Omega_{2q}$, $q \geq 2$. At last, in Section 5, we make a brief discussion on the connection between positive definite functions on $\Omega_{2q}$ and on real spheres.

### 2. PRELIMINARY AND BASIC LEMMATA

The Laplace-Beltrami operator $\Delta_{2q}$ associated to the Bergman metric on $B_{2q}$ reveals itself in the field of several complex variables and is the basic invariant differential operator on the symmetric space isomorphic to $B_{2q}$. It can be defined by

$$
\Delta_{2q} = \frac{4}{q+1} \left(1 - |z|^2\right) \sum_{\mu, \nu=1}^{q} (\delta_{\mu\nu} - z_{\mu} \overline{z_{\nu}}) \frac{\partial^2}{\partial z_{\mu} \partial \overline{z_{\nu}}}, \quad z = (z_1, z_2, \ldots, z_q) \in B_{2q},
$$
and it is usually attached to the Poisson-Szegő kernel on $\Omega_{2q}$, i.e., the kernel
\[
P_q(z, \zeta) := \frac{(q - 1)!}{2\pi^q} \frac{(1 - |z|^2)^q}{|1 - \langle z, \zeta \rangle|^{2q}}, \quad (z, \zeta) \in B_{2q} \times \Omega_{2q}.
\]
Together, they appear as the main subject in the following Dirichlet problem ([12]):

**Lemma 2.1.** Given a continuous function $f : \Omega_{2q} \to \mathbb{C}$, there exists a continuous function $u : B_{2q} \to \mathbb{C}$ such that $\Delta_{2q} u = f$ and $u|_{\Omega_{2q}} = f$. The solution $u$ can be computed by the formula
\[
u(z) = \int_{\Omega_{2q}} P_q(z, \zeta) f(\zeta) d\omega_{2q}(\zeta), \quad z \in B_{2q},
\]
in which $d\omega_{2q}$ denotes the rotation-invariant surface element on $\Omega_{2q}$.

Some additional properties of the Poisson-Szegő kernel are: $P_q \geq 0$, $P_q(\cdot, \zeta)$ is annihilated by $\Delta_{2q}$ and $P_q$ coincides with the standard Poisson kernel when $q = 1$. If inner product between square-integrable functions on $\Omega_{2q}$ is defined by
\[
\langle f, g \rangle = \int_{\Omega_{2q}} f(\zeta) \overline{g(\zeta)} d\omega_{2q}(\zeta),
\]
then the following orthogonal decomposition holds
\[
L^2(\Omega_{2q}, d\omega_{2q}) = \bigoplus_{m,n=0}^{\infty} H^q_{m,n},
\]
in which $H^q_{m,n}$ is the space of restrictions to $\Omega_{2q}$ of harmonic polynomials in the variables $z$ and $\overline{z}$ on $\mathbb{C}^q$, that are homogeneous of degree $m$ in $z$ and degree $n$ in $\overline{z}$. Here, a harmonic function is understood to be one that is annihilated by the complex Laplacian
\[
\sum_{j=1}^{q} \frac{\partial^2}{\partial z_j \overline{z_j}}, \quad z = (z_1, z_2, \ldots, z_q).
\]

Several classes of disk polynomials play an important role in the spaces just described. Recall that the disk polynomial of degree $m+n$ in $x$ and $y$ associated to a positive real number $\alpha$ is the polynomial $R_{m,n}^\alpha$ given by
\[
R_{m,n}^\alpha(z) := r^{m-n} e^{i(m-n)\theta} R_{m \wedge n}^{(\alpha, |m-n|)}(2r^2 - 1), \quad z = re^{i\theta} = x + iy,
\]
in which $R_{m \wedge n}^{(\alpha, |m-n|)}$ is the usual Jacobi polynomial of degree $m \wedge n := \min\{m, n\}$ associated to the numbers $\alpha$ and $|m-n|$, and normalized by $R_{m \wedge n}^{(\alpha, |m-n|)}(1) = 1$. It is easy to see that $R_{m,n}^\alpha$ is a polynomial of degree $m$ in the variable $z$ and of degree $n$ in the variable $\overline{z}$. Due
to the orthogonality relations for Jacobi polynomials, the set \( \{ R_{m,n}^q : 0 \leq m, n < \infty \} \) is a complete orthogonal system in \( L^2(B_2, dw_\alpha) \), where \( dw_\alpha \) is the positive measure of total mass one on \( B_2 \) given by

\[
dw_\alpha(z) = \frac{\alpha + 1}{\pi} (1 - x^2 - y^2)^\alpha \, dx dy, \quad z = x + iy.
\]

For further information on disk polynomials see [2,4,9] and several references given there.

The isomorphism between \( \Omega_{2q} \) and the homogeneous space \( U(q)/U(q-1) \), the quotient of the group \( U(q) \) of unitary transformations of \( \mathbb{C}^q \) by its subgroup \( U(q-1) \) composed of unitary transformations that leave a “pole” of \( \Omega_{2q} \) fixed, is the doorway for a beautiful theory in which disk polynomials play the role played by ultraspherical polynomials in harmonic analysis on real spheres ([1,8,9]). The connection between disk polynomials and the spaces \( \mathcal{H}_q^{m,n} \) is made by the following result implicit in [9].

**Lemma 2.2.** The \( \zeta \)-zonal functions in \( \mathcal{H}_q^{m,n} \), \( q \geq 2 \), are the constant multiples of the function \( \xi \in \Omega_{2q} \rightarrow R_{m,n}^{-2}(\langle \xi, \zeta \rangle) \).

If \{\( Y_1^q, Y_2^q, \ldots, Y_{N(q,m,n)}^q \)\} is an orthonormal basis for \( \mathcal{H}_q^{m,n} \) with respect to the inner product previously defined then the following nice addition formula holds ([9]):

\[
R_{m,n}^{-2}(\langle \xi, \zeta \rangle) = \frac{\omega_{2q}}{N(q; m, n)} \sum_{j=1}^{N(q;m,n)} Y_j^q(\xi)Y_j^q(\zeta), \quad \xi, \zeta \in \Omega_{2q}.
\]

In this expression, \( \omega_{2q} \) is the total surface of \( \Omega_{2q} \), i.e., \( \omega_{2q} = 2\pi^q/(q-1)! \). Among other things this formula reveals that \( R_{m,n}^{-2} \) is positive definite on \( \Omega_{2q} \). Indeed, by the use of the addition formula we have that

\[
\sum_{\mu=1}^{l} \sum_{\nu=1}^{l} c_{\mu}c_{\nu} R_{m,n}^{-2}(\langle \xi_\mu, \xi_\nu \rangle) = \sum_{\mu=1}^{l} \sum_{\nu=1}^{l} c_{\mu}c_{\nu} \frac{\omega_{2q}}{N(q; m, n)} \sum_{j=1}^{N(q;m,n)} Y_j^q(\xi_\mu)Y_j^q(\xi_\nu)
\]

\[
= \frac{\omega_{2q}}{N(q; m, n)} \sum_{j=1}^{N(q;m,n)} \sum_{\mu=1}^{l} c_{\mu} Y_j^q(\xi_\mu) \sum_{\nu=1}^{l} c_{\nu} Y_j^q(\xi_\nu)
\]

\[
= \frac{\omega_{2q}}{N(q; m, n)} \sum_{j=1}^{N(q;m,n)} \left| \sum_{\mu=1}^{l} c_{\mu} Y_j^q(\xi_\mu) \right|^2 \geq 0,
\]

whenever \( l \geq 1 \), \( \{\xi_1, \xi_2, \ldots, \xi_l\} \subset \Omega_{2q} \) and \( \{c_1, c_2, \ldots, c_l\} \subset \mathbb{C} \).

The last property of disk polynomials needed in this paper is the main result in [5], a series representation for the Poisson-Szegő kernel in terms of disk polynomials.

**Lemma 2.3.** If \( q \geq 2 \) then

\[
\mathcal{P}_q(r\xi, \zeta) = \sum_{(m,n) \in \mathbb{N}^2} \frac{N(q; m, n)}{\omega_{2q}} S_{m,n}^q(r) R_{m,n}^{-2}(\langle \xi, \zeta \rangle), \quad 0 \leq r < 1, \quad \xi, \zeta \in \Omega_{2q},
\]

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in which \( S_{m,n}(r) \geq 0 \) and \( \lim_{r \to 1^{-}} S_{m,n}(r) = 1 \). The series converges absolutely and uniformly for \( \xi, \zeta \in \Omega_{2q} \) and \( 0 \leq r \leq \rho \), for each \( \rho < 1 \).

We observe that the function \( S_{m,n}(r) \) is in fact a multiple of the product of \( r^{m+n} \) by the hypergeometric function \( F(m,n,m+n+q,r^2) \) associated to the triple \( (m,n,m+n+q) \), as explained in [5].

We end this section with a consequence of the definition of positive definiteness on spheres to be used in the next two sections. The first step to reach it is to obtain a complex version of a well-known result found by W. H. Young a long time ago ([14]).

**Lemma 2.4** A continuous function \( f \) is positive definite on \( \Omega_{2q} \) only if

\[
\int_{\Omega_{2q}} \int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) g(\xi) \overline{g(\zeta)} d\omega_{2q}(\xi) d\omega_{2q}(\zeta) \geq 0,
\]

for every continuous function \( g : \Omega_{2q} \to \mathbb{C} \).

**Proof.** Let \( f \) be positive definite on \( \Omega_{2q} \) and \( g \) a continuous function on \( \Omega_{2q} \). Since the inequality

\[
\sum_{\mu, \nu = 1}^{l} c_{\mu} \overline{c_{\nu}} f(\langle \xi_{\mu}, \xi_{\nu} \rangle) \geq 0,
\]

holds for every positive integer \( l \), \( \{\xi_{1}, \xi_{2}, \ldots, \xi_{l}\} \subset \Omega_{2q} \) and \( \{c_{1}, c_{2}, \ldots, c_{l}\} \subset \mathbb{C} \), we take \( l \geq 2 \), \( c_{\mu} = g(\xi_{\mu}) \), \( \mu = 1, 2, \ldots, l \), and integrate with respect to \( \xi_{\mu} \) and \( \xi_{\nu} \). From each of the diagonal terms we obtain

\[
\omega_{2q} f(1) \int_{\Omega_{2q}} |g(\xi_{\mu})|^2 d\omega_{2q}(\xi_{\mu}),
\]

while each of the terms for which \( \mu \neq \nu \) yields

\[
\int_{\Omega_{2q}} \int_{\Omega_{2q}} f(\langle \xi_{\mu}, \xi_{\nu} \rangle) g(\xi_{\mu}) \overline{g(\xi_{\nu})} d\omega_{2q}(\xi_{\mu}) d\omega_{2q}(\xi_{\nu}).
\]

Thus,

\[
l \omega_{2q} f(1) \int_{\Omega_{2q}} |g(\xi)|^2 d\omega_{2q}(\xi) + l(l - 1) \int_{\Omega_{2q}} \int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) g(\xi) \overline{g(\zeta)} d\omega_{2q}(\xi) d\omega_{2q}(\zeta) \geq 0.
\]

The continuity of \( g \) implies that the single integral above is finite. Hence, on dividing by \( l(l - 1) \) and letting \( l \) goes to infinity, we deduce that

\[
\int_{\Omega_{2q}} \int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) g(\xi) \overline{g(\zeta)} d\omega_{2q}(\xi) d\omega_{2q}(\zeta) \geq 0,
\]

which is the desired inequality.

We shall use the following weak version of the previous lemma.
Corollary 2.5  A continuous function \( f \) is positive definite on \( \Omega_{2q} \) only if

\[
\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) \, d\omega_{2q}(\xi) \geq 0, \quad \zeta \in \Omega_{2q}.
\]

Proof. First, take \( g \) equals to a constant in the previous theorem to conclude that a positive definite function \( f \) on \( \Omega_{2q} \) satisfies

\[
\int_{\Omega_{2q}} \int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) \, d\omega_{2q}(\xi) \, d\omega_{2q}(\zeta) \geq 0.
\]

Next, observe that to conclude the proof it suffices to show that the following invariance property holds:

\[
\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) \, d\omega_{2q}(\xi) = \int_{\Omega_{2q}} f(\langle \xi, \eta \rangle) \, d\omega_{2q}(\xi), \quad \zeta, \eta \in \Omega_{2q}.
\]

Since this is certainly true in the case \( q = 1 \), we assume that \( q \geq 2 \). By decomposing the point \( \xi \) in the form

\[
\xi = \cos \phi \, e^{i\varphi} \zeta + \sin \phi \, \xi', \quad \xi' \in \Omega_{2q-2}, \quad \phi \in (0, 2\pi), \quad \varphi \in \mathbb{R} \mod 2\pi,
\]

we see that

\[
d\omega_{2q}(\xi) = \cos \phi (\sin \phi)^{2q-3} d\phi d\varphi d\omega_{2q-2}(\xi').
\]

Making the changes \( r = \cos \phi \) and \( \varphi = -\theta \), we reach

\[
d\omega_{2q}(\xi) = r(1-r^2) dr d\theta d\omega_{2q-2}(\xi'),
\]

and, consequently,

\[
\int_{\Omega_{2q}} f(\langle \xi, e^{i\varphi} \zeta \rangle) \, d\omega_{2q}(\xi) = 2\pi \omega_{2q-2} \int_{-1}^{1} \int_{-r}^{r} r(1-r^2)^{q-2} f(r) \, dr,
\]

which is independent of \( \varphi \) and \( \zeta \). Finally, taking an element \( B \) of \( U(q) \) such that \( B(e^{i\varphi} \zeta) = \zeta \) and using the invariance of \( d\omega_{2q} \) with respect to elements of \( U(q) \), we obtain

\[
\int_{\Omega_{2q}} f(\langle \xi, \zeta \rangle) \, d\omega_{2q}(\xi) = \int_{\Omega_{2q}} f(\langle \xi, e^{i\varphi} \zeta \rangle) \, d\omega_{2q}(\xi).
\]

The result follows.
3. ZONAL POSITIVE DEFINITE KERNELS ON $\Omega_2$

The starting point in this section is the Poisson-Szegő kernel on $\Omega_2$. Since it coincides with the Poisson kernel, we have ([12]):

$$P_1(z, \zeta) = \frac{1 - |z|^2}{2\pi|1 - (z, \zeta)|^2}, \quad (z, \zeta) \in B_2 \times \Omega_2.$$  

Writing $z$ in polar form the previous equation reads

$$P_1(r\xi, \zeta) = \sum_{k \in \mathbb{Z}} r^{|k|} \langle \xi, \zeta \rangle^k, \quad \xi, \zeta \in \Omega_2.$$  

The solution of the Dirichlet problem on $\Omega_2$ given in Lemma 2.1 can now be expressed in the form

$$\int_{\Omega_2} P_1(r\xi, \zeta)f(\zeta)d\omega_2(\zeta) = \sum_{k \in \mathbb{Z}} r^{|k|} \int_{\Omega_2} f(\zeta)\langle \xi, \zeta \rangle^k d\omega_2(\zeta), \quad \xi, \zeta \in \Omega_2.$$  

Recalling Lemma 2.1, we now see that a continuous function $f : \Omega_2 \to \mathbb{C}$ has a series representation in the form

$$f(\xi) \sim \sum_{k \in \mathbb{Z}} \int_{\Omega_2} f(\zeta)\langle \xi, \zeta \rangle^k d\omega_2(\zeta), \quad \xi, \zeta \in \Omega_2.$$  

The use of this representation and of Corollary 2.5 yield our next result.

**Theorem 3.1** A continuous function $f$ is positive definite on $\Omega_2$ if and only if it is representable in the form

$$f(\xi) = \sum_{k \in \mathbb{Z}} a_k \xi^k, \quad \xi \in \Omega_2,$$

in which $\sum_{k \in \mathbb{Z}} a_k < \infty$ and $a_k \geq 0$ for all $k$.

**Proof.** Let $f$ be positive definite on $\Omega_2$. For $\eta \in \Omega_2$, the function $f(\langle \cdot, \eta \rangle)$ can be represented in the form

$$f(\langle \xi, \eta \rangle) \sim \sum_{k \in \mathbb{Z}} \int_{\Omega_2} f(\langle \zeta, \eta \rangle)\langle \eta, \zeta \rangle^k \langle \xi, \eta \rangle^k d\omega_2(\zeta)$$

$$= \sum_{k \in \mathbb{Z}} \left( \int_{\Omega_2} f(\langle \zeta, \eta \rangle)\langle \eta, \zeta \rangle^k d\omega_2(\zeta) \right) \langle \xi, \eta \rangle^k, \quad \xi \in \Omega_2.$$  

The function $\xi \in \Omega_2 \to f(\xi)\overline{\xi}^k$ is positive definite on $\Omega_2$ because it is a product of positive definite functions there. By Corollary 2.5, the integral

$$\int_{\Omega_2} f(\langle \zeta, \eta \rangle)\langle \eta, \zeta \rangle^k d\omega_2(\zeta)$$

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is nonnegative and independent of $\eta$. Thus, we can write

$$f(\langle \xi, \eta \rangle) \sim \sum_{k \in \mathbb{Z}} a_k \langle \xi, \eta \rangle^k, \quad a_k \geq 0, \quad \xi, \eta \in \Omega_2.$$

In particular, we can write

$$f(\xi) \sim \sum_{k \in \mathbb{Z}} a_k \xi^k, \quad a_k \geq 0, \quad \xi \in \Omega_2.$$

To prove $f$ coincides with the series it suffices to show that the series is absolutely convergent on $\Omega_2$. But, letting $r \to 1^-$ in the inequality

$$\sum_{k=-m}^{m} a_k r^{|k|} |\xi| \leq \sum_{k \in \mathbb{Z}} a_k r^{|k|}, \quad m \in \mathbb{N},$$

and using Lemma 2.1, we obtain

$$\sum_{k=-m}^{m} a_k |\xi| \leq \sum_{k=-m}^{m} a_k \leq \lim_{r \to 1^-} \sum_{k \in \mathbb{Z}} a_k r^{|k|} = f(1), \quad m \in \mathbb{N}.$$

Conversely, let $f$ have a series representation as in the statement of the theorem. Taking an arbitrary positive integer $l$, complex numbers $c_1, c_2, \ldots, c_l$ and $\xi_1, \xi_2, \ldots, \xi_l$ in $\Omega_2$ we have that

$$\sum_{\mu=1}^{l} \sum_{\nu=1}^{l} c_{\mu} \overline{c_{\nu}} f(\langle \xi_\mu, \xi_\nu \rangle) = \sum_{k \in \mathbb{Z}} a_k \sum_{\mu=1}^{l} \sum_{\nu=1}^{l} c_{\mu} \overline{c_{\nu}} \langle \xi_\mu, \xi_\nu \rangle^k$$

$$= \sum_{k \in \mathbb{Z}} a_k \left( \sum_{\mu=1}^{l} c_{\mu} \xi_\mu, \sum_{\nu=1}^{l} c_{\nu} \xi_\nu \right)^k$$

$$= \sum_{k \in \mathbb{Z}} a_k \left| \sum_{\mu=1}^{l} c_{\mu} \xi_\mu \right|^{2k} \geq 0.$$

Therefore, $f$ is positive definite on $\Omega_2$. \hfill \blacksquare

4. POSITIVE DEFINITE KERNELS ON $\Omega_{2q}$, $Q \geq 2$

In this section, we generalize Theorem 3.1 by characterizing the positive definite functions on $\Omega_{2q}$, $q \geq 2$. For a continuous function $f : \Omega_{2q} \to \mathbb{C}$ we consider once again the solution of the Dirichlet problem on $B_{2q}$, i.e., we define

$$f(r\xi) := \int_{\Omega_{2q}} \mathcal{P}_{q}(r\xi, \eta) f(\eta) d\omega_{2q}(\eta), \quad 0 \leq r < 1, \quad \xi \in \Omega_{2q}.$$
Multiplying the equation in Lemma 2.3 by $f(\eta)$ and integrating we reach
\[
\int_{\Omega_2 q} P_q(r\xi, \eta)f(\eta)d\omega_{2q}(\eta) = \sum_{(m,n)\in\mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} S_{m,n}^q(r) \int_{\Omega_2 q} f(\eta) R_{m,n}^{-2}(\langle \xi, \eta \rangle) d\omega_{2q}(\eta),
\]
i.e.,
\[
f(r\xi) = \sum_{(m,n)\in\mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} S_{m,n}^q(r) \int_{\Omega_2 q} f(\eta) R_{m,n}^{-2}(\langle \xi, \eta \rangle) d\omega_{2q}(\eta), \quad 0 \leq r < 1, \quad \xi \in \Omega_{2q}.
\]
Thus, on $\Omega_{2q}$, we can write
\[
f(\xi) \sim \sum_{(m,n)\in\mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} \int_{\Omega_2 q} f(\eta) R_{m,n}^{-2}(\langle \xi, \eta \rangle) d\omega_{2q}(\eta).
\]
Next, we analyze the above identification for the case in which $f$ is $\zeta$-zonal. We define
\[
\Psi(\xi) := \int_{\Omega_2 q} f(\langle \eta, \zeta \rangle) R_{m,n}^{-2}(\langle \xi, \eta \rangle) d\omega_{2q}(\eta), \quad \xi \in \Omega_{2q},
\]
and observe that $\Psi$ is also $\zeta$-zonal. Indeed, to see that it suffices ([9]) to verify that $\Psi \circ B = \Psi$ for every unitary transformation $B$ of $\mathbb{C}^q$ fixing $\zeta$. But, for any such $B$,
\[
\Psi(B\xi) = \int_{\Omega_2 q} f(\langle \eta, B\zeta \rangle) R_{m,n}^{-2}(\langle B\xi, \eta \rangle) d\omega_{2q}(\eta)
\]
\[
= \int_{\Omega_2 q} f(\langle B^*\eta, \zeta \rangle) R_{m,n}^{-2}(\langle B^*\xi, B^*\eta \rangle) d\omega_{2q}(B^*\eta)
\]
\[
= \int_{\Omega_2 q} f(\langle \eta, \zeta \rangle) R_{m,n}^{-2}(\langle \xi, \eta \rangle) d\omega_{2q}(\eta) = \psi(\xi), \quad \xi \in \Omega_{2q}.
\]
Since the function $\xi \mapsto R_{m,n}^{-2}(\langle \xi, \zeta \rangle)$ is an element of $\mathcal{H}_{m,n}^q$ so is $\Psi$. Due to Lemma 2.2, we conclude that
\[
\Psi(\xi) = M P_{m,n}^{q-2,|m-n|}(1) R_{m,n}^{-2}(\langle \xi, \zeta \rangle), \quad \xi \in \Omega_{2q},
\]
for some constant $M$. The fact that $R_{m,n}^{-2}(1) = 1$ allows us to infer that
\[
\Psi(\zeta) = M P_{m,n}^{q-2,|m-n|}(1).
\]
Thus, we have proved the following result.

**Lemma 4.1.** For a continuous function $f$ on $B_2$ and $\xi, \zeta$ on $\Omega_{2q}$, the following expansion holds:
\[
f(\langle \xi, \zeta \rangle) \sim \sum_{(m,n)\in\mathbb{N}^2} \frac{N(q;m,n)}{\omega_{2q}} \left( \int_{\Omega_2 q} f(\langle \eta, \zeta \rangle) R_{m,n}^{-2}(\langle \xi, \eta \rangle) d\omega_{2q}(\eta) \right) R_{m,n}^{-2}(\langle \xi, \zeta \rangle).
\]
We are now ready to prove the main result of this section.

**Theorem 4.2.** A continuous function $f$ is positive definite on $\Omega_{2q}$ if and only if it is of the form

$$f(z) = \sum_{(m,n) \in \mathbb{N}^2} a_{m,n} R_{m,n}^{q-2}(z), \quad z \in B_2,$$

in which $\sum_{(m,n) \in \mathbb{N}^2} a_{m,n} < \infty$ and $a_{m,n} \geq 0$ for all $(m,n)$.

**Proof.** First, assume that $f$ is positive definite on $\Omega_{2q}$. By Lemma 4.1, we may write

$$f(\langle \xi, \zeta \rangle) \sim \sum_{(m,n) \in \mathbb{N}^2} \frac{N(q; m,n)}{\omega_{2q}} a_{m,n}^q(\zeta) R_{m,n}^{q-2}(\langle \xi, \zeta \rangle), \quad \xi, \zeta \in \Omega_{2q},$$

in which

$$a_{m,n}^q(\zeta) := \int_{\Omega_{2q}} f(\langle \eta, \zeta \rangle) R_{m,n}^{q-2}(\langle \zeta, \eta \rangle) d\omega_{2q}(\eta).$$

By Corollary 2.5, $a_{m,n}^q$ is a nonnegative constant and, therefore, we can write

$$f(\langle \xi, \zeta \rangle) \sim \sum_{(m,n) \in \mathbb{N}^2} a_{m,n} R_{m,n}^{q-2}(\langle \xi, \zeta \rangle), \quad a_{m,n} \geq 0, \quad \xi, \zeta \in \Omega_{2q}.$$

To see that $f(\langle \xi, \zeta \rangle)$ coincides with the series, it suffices to see that

$$\sum_{k=1}^{\infty} \sum_{(m,n) \in \mathbb{N}^2} a_{m,n} S_{m,n}^q(r) R_{m,n}^{q-2}(\langle \xi, \zeta \rangle) \omega_{2q} \leq \sum_{k=1}^{\infty} \sum_{(m,n) \in \mathbb{N}^2} a_{m,n} S_{m,n}^q(r) R_{m,n}^{q-2}(1)$$

$$\leq \sum_{(m,n) \in \mathbb{N}^2} a_{m,n} S_{m,n}^q(r) R_{m,n}^{q-2}(1), \quad 0 \leq r < 1, \quad k \in \mathbb{N}.$$
whenever \( l \geq 1 \), \( \{\xi_1, \xi_2, \ldots, \xi_l\} \subset \Omega_{2q} \) and \( \{c_1, c_2, \ldots, c_l\} \subset \mathbb{C} \). Thus, \( f \) is positive definite on \( \Omega_{2q} \).

5. ADDITIONAL CONSIDERATIONS

In this final section, we discuss one aspect on the connection between positive definiteness on \( \Omega_{2q} \) and positive definiteness on real spheres. Let us denote by \( S^q \) the unit sphere in \( \mathbb{R}^{q+1} \). Since \( \Omega_{2q} \) contains a copy of \( S^{q-1} \), it follows at once that every positive definite function on \( \Omega_{2q} \) is positive definite on \( S^{q-1} \). What does not seem to be so obvious is that a positive definite function on \( \Omega_{2q} \), \( q \geq 2 \), can be positive definite on \( S^m \), \( q \leq m \leq 2q-1 \).

In order to explain that this can happen, we need the following result linking ultraspherical polynomials and disk polynomials discovered by Dreseler and Hrach ([4]).

**Lemma 5.1.** For any \( \alpha \geq 0 \) and any nonnegative integer \( k \), there are positive constants \( \{d(k, \alpha, m, n) : m + n = k\} \) such that

\[
R_k^{(\alpha+1/2, \alpha+1/2)}(Re z) = \sum_{m+n=k} d(k, \alpha, m, n) R_{m,n}^\alpha(z), \quad z \in B_2.
\]

We observe that the coefficients \( d(k, \alpha, m, n) \) can be computed by the formula

\[
d(k, \alpha, m, n) = \left( \int_{-1}^1 \left| R_k^{(\alpha+1/2, \alpha+1/2)}(x) \right|^2 dw_{\alpha+1/2, \alpha+1/2}(x) \right) \left( \int_{B_2} \left| R_{m,n}^\alpha(z) \right|^2 dw_{\alpha}(z) \right)^{-1},
\]

where \( dw_{\alpha+1/2, \alpha+1/2}(x) = c(\alpha)(1-x^2)^{\alpha+1/2}dx \) and \( c(\alpha) \) is a constant chosen so that the measure has total mass equals one.

Let \( q : [-1, 1] \rightarrow \mathbb{R} \) be a positive definite function on \( S^{2q-1} \), \( q \geq 2 \). According to Schoenberg’s theorem,

\[
g(x) = \sum_{k \in K} a_k R_k^{(q-3/2, q-3/2)}(x), \quad K \subset \mathbb{N}, \quad a_k \geq 0, \quad \sum_{k \in K} a_k < \infty.
\]

By Lemma 5.1, we obtain

\[
g(Re z) = \sum_{k \in K} a_k \sum_{m+n=k} d(k, q-2, m, n) R_{m,n}^{q-2}(z)
= \sum_{k \in K} a_k \sum_{m+n=k} d(k, q-2, m, n) R_{m,n}^{q-2}(z)
= \sum_{m+n \in K} a_{m+n} d(m+n, q-2, m, n) R_{m,n}^{q-2}(z), \quad z \in B_2.
\]

Due to the assumptions on \( g \), the resulting sum defines a real positive definite function \( f \) on \( \Omega_{2q} \). Since \( g \) is the restriction of \( f \) to \([-1, 1]\), it follows that \( f \) is positive definite on
Thus, we have concluded that there are positive definite functions on \( \Omega_{2q} \) that are positive definite not only on \( S^{q-1} \) but also on all real spheres of dimension up to \( 2q \).

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