Remarks on Immersions in the Metastable Range Dimension

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Let $f : M \to N$ be a continuous map between two closed manifolds such that $f_* : H_*(M, \mathbb{Z}_2) \to H_*(N, \mathbb{Z}_2)$ is an isomorphism. Suppose that $N$ immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$. Then $M$ also immerses in $\mathbb{R}^{n+k}$. We use techniques of normal bordism theory to prove this result which complements the work Biasi et al. ([BGL]). We also present conditions for a map to be homotopic to an immersion.

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1. INTRODUCTION

This paper is concerned about conditions which on there exist immersions in the metastable range dimension, which were already considered in [BGL]. For the two problems we have considered, we give conditions on the induced maps in homology groups with $\mathbb{Z}_2$ coefficients.

For the first problem, let us consider $h : M^n \to X^{n+k}$ a continuous map from a closed smooth connected $n$-manifold into a smooth connected $(n+k)$-manifold, $5 \leq n < 2k$, bordant to an immersion, in the sense of Conner and Floyd. When is $h$ homotopic to an immersion? By exploiting the works of Koschorke ([K1],[K2],[K3]) and Salomonsen ([S]) we obtain an answer in terms of the induced homology groups maps.

For the second problem, let $f : M \to N$ be a continuous map between two closed smooth connected $n$-dimensional manifolds. Suppose that $N$ immerses in $\mathbb{R}^{n+k}$, for some $k$, with $5 \leq n < 2k$. Under which conditions on $f$ does $M$ immerses in $\mathbb{R}^{n+k}$? The case when it is supposed that $M$ immerses in $\mathbb{R}^{n+k}$ and it is looking for conditions on $f$ such that $N$ also immerses in $\mathbb{R}^{n+k}$ has been considered in the work of Biasi et al. ([BGL]) and Glover et al. ([GH1],[GH2],[GM1]).
We will use a normal bordism approach to investigate these problems. We prove the following main results:

**Theorem A:** Let \( h : M^n \to X^{n+k} \) be a continuous map from a closed smooth connected \( n \)-manifold into a smooth connected \((n+k)\)-manifold, \( 5 \leq n < 2k \), and let \( g : M \to BO(q) \), for \( q \) large, be the classifying map of \( \nu_h \), the stable normal bundle of \( h \). Suppose that

\[ (h, g) : M \to X \times BO(q), \quad q \text{ large}, \]

induces

\[ (h, g)_* : H_i(M, \mathbb{Z}_2) \to H_i(X \times BO(q), \mathbb{Z}_2) \]

which is an isomorphism for \( i < n-k \) and an epimorphism, for \( i = n-k \).

Then if \( h \) is bordant to an immersion, \( h \) is homotopic to an immersion.

**Theorem B:** Let \( M \) and \( N \) be a closed connected \( n \)-manifolds and let \( f : M \to N \) be a continuous map such that

\[ f_* : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2) \]

is an isomorphism, for \( i \geq 0 \).

Then if \( N \) immerses in \( \mathbb{R}^{n+k} \) for \( 5 \leq n < 2k \), so does \( M \).

The work is divided in four sections. In Section 2, we present two exact sequences of bordism groups. One of them is a generalization of the exact sequence normal bordism group given by Salomonsen [S], by using identifications of some normal bordism groups.

In Section 3, we prove Theorem A and B and in Section 4, we present an application using a non standard obstruction theory.

**Notations and conventions:** In this work, \( C \) will denote the class of all torsion groups where the torsion is odd.

### 2. EXACT SEQUENCES OF BORDISM GROUPS

Given a topological space \( X \) and a virtual bundle \( \phi, \Omega_i(X, \phi) \) denotes the \( i \)-th normal bordism group of \( X \) with coefficient \( \phi \). For the definition and more details about normal bordism see [K1] or [S]. We adopt the Salomonsen convention.

We get a generalization of the exact sequence normal bordism group given by Salomonsen [S], by using identifications of some normal bordism groups.

For each \( q \), let \( \varphi^q = \varepsilon^{p+q} - (\alpha^p \times \gamma^q) \) and \( \psi^q = \gamma^q - \varepsilon^q \) be virtual bundles over \( X \times BO(q) \), where \( \gamma^q \) denotes the universal vector bundle over \( BO(q) \). We can construct a fibre bundle \( \tilde{V}_k(\psi^q) \to X \) with \((k-1)\)-connected fibre and we define a \( k \)-dimensional vector bundle \( \mu^k \) over \( \tilde{V}_k(\psi^q) \) such that \( \mu^k \oplus \varepsilon^q \simeq \varepsilon^k \oplus \gamma^q \). ([S]).

Let us consider \( \theta' : \tilde{V}_k(\psi^q) \to BO(k) \) the classifying map of the vector bundle \( \mu^k \), which is a high homotopy equivalence, for \( k \) large enough.

The following diagram is commutative:
where $\theta_*$, induced by $\theta'$, is an isomorphism for $q$ large.

Also, for $q$ large, $\Omega_n(X \times BO(q), \varphi^q) \simeq \pi_n^{S \hat{\pi} + q}(T(\alpha) \wedge MO(q))$, where $T(\alpha)$ is the Thom space ([K1]). Since $T(\alpha)$ is $(p - 1)$-connected we can conclude that $\eta_n(X) \simeq \Omega_n(X \times BO(\xi), \varphi^q)$ and this normal bordism group does not depend of $\alpha^p$.

Let us denote by $I_n(X)$ the bordism group of continuous maps $h : M^n \to X^{n+k}$, which are homotopic to an immersion and let $\mathcal{F} : I_n(X) \to \eta_n(X)$ be the forgetful map.

Let us consider $X$ a $(n + k)$-manifold and let $\nu^p_X$ be the stable normal bundle of $X$. If $\varphi^k = \varepsilon^{p+k} - \nu^p_X \times \gamma^k$, for $p$ large, an element of $\Omega_n(X \times BO(k), \varphi^k)$ can be considered as $[M^n, (h, g), H]$ where $(h, g) : M^n \to X \times BO(k)$ is a continuous map,

$$H : TM \oplus h^*(\nu_X^p) \oplus g^*(\gamma^k) \to \varepsilon^{p+k} \oplus \varepsilon^n$$

is a stable isomorphism and $g$ is the classifying map of the stable normal bundle $\nu_X^p$. Therefore, $\Omega_n(X \times BO(k), \varphi^k)$ can be identified with $I_n(X)$.

By using these identifications and diagram (I), in Salomonsen sequence ([S]) we get the following exact sequence, for $q$ large and $n \leq 2k + 2$.

$$\begin{array}{ccc}
(II) & \longrightarrow & \Omega_{n-k}(X \times BO(q) \times P^\infty, \Gamma_k) \longrightarrow I_n(X) \xrightarrow{\mathcal{F}} \eta_n(X) \xrightarrow{\tilde{\gamma}_{k-1}} \\
& \longrightarrow & \Omega_{n-k-1}(X \times BO(q) \times P^\infty, \Gamma_{k-1}) \longrightarrow \ldots
\end{array}$$

where $\Gamma_k = \nu_X^p \times \gamma^q \oplus (\varepsilon^{q-k} - \gamma^q) \otimes \lambda - \varepsilon^{p+q+n-k}$ and $\lambda$ is the canonical line bundle over the real projective space $P^\infty$.

Let us consider now $\psi$ a virtual vector bundle over $M$.

From the exact sequence of Salomonsen, for $5 \leq n < 2k$, we have the following exact sequence:

$$\begin{array}{ccc}
(III) & \longrightarrow & \Omega_n(\tilde{V}_k(\psi), TM^0) \xrightarrow{\pi_M} \Omega_n(M \times TM^0) \xrightarrow{\gamma_M} \Omega_{n-k-1}(M \times P^\infty, \Phi) \longrightarrow \ldots
\end{array}$$

where $\Phi = -(n - k - 1)\lambda - \lambda \otimes \psi + TM^0$ and $\gamma_M$ is defined by the construction of the sequence.

We recall that if $\psi = h^*TX - \varepsilon^k \oplus TM$, then $\gamma_M([M])$ is the invariant $\omega_\xi(\nu_h)$ defined by Koschorke ([K2]), ([K3]), which is an obstruction to the existence of a monomorphism from $M \times \mathbb{R}^k$ into $\gamma_h$. 

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Here, $[M] = [M,1_M,t_M] \in \Omega_n(M,TM^0)$ is the fundamental class of $M$ where $t_M : TM \oplus \varepsilon^n \to \varepsilon^n \oplus TM$ is the isomorphism which interchange factors.

3. PROOFS OF THEOREMS A AND B

Proof of Theorem A:
Let $h : M \to X$ be a continuous map from a closed connected smooth $n$-dimensional manifold $M$ into a smooth connected $(n+k)$-dimensional manifold $X$.

Let us consider now the following commutative diagram, where the left hand vertical sequence is (III) and the right hand vertical sequence is (II) and $(h,g)_*$ and $((h,g) \times Id)_*$ are induced maps of $(h,g)$ in convenient normal bordism groups.

\[ \begin{array}{ccc}
\Omega_n(M,TM^0) & \xrightarrow{(h,g)_*} & \gamma_n(X) \\
O_{n-k-1}(M \times \mathbb{P}^\infty, \Phi) & \xrightarrow{((h,g) \times Id)_*} & O_{n-k-1}(X \times BO(q) \times \mathbb{P}^\infty, \Gamma_{k-1}) \\
\end{array} \]

Suppose that $h$ is bordant to an immersion. Then

\[ 0 = \tilde{\gamma}_{k-1}([M,h]) = ((h,g) \times Id)_* (\gamma_M([M])) . \]

Since $(h,g) : M \to X \times BO(q)$, $q$ large, induces

\[ (h,g)_* : H_i(M,\mathbb{Z}_2) \to H_i(X \times BO(q),\mathbb{Z}_2) \]

which is an isomorphism for $i < n-k$ and an epimorphism for $i = n-k$, we conclude that $((h,g) \times Id)_*$ is a $C$-isomorphism for $i = n-k-1$ and then ker($((h,g) \times Id)_*$) $\in C$.

We recall that the elements of image of $\gamma_M$ have order a potency of 2. Therefore $\gamma_M([M,h]) = 0$ and $h$ is homotopic to an immersion.

Proof of Theorem B:
We recall that under hypotheses of Theorem B,

\[ f_* : \Omega_n(M,f^*TN^0) \to \Omega_n(N,TN^0) \]

is a $C$-isomorphism and $f^*(\beta_2) = \alpha_2$, where $\alpha = \nu_M$ and $\beta = \nu_N$ are the stable normal bundles of $M$ and $N$ and $\alpha_2$ and $\beta_2$, the respectively 2-localization ([BGL]).

Let us consider the following commutative diagram
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\[\Omega_n(\tilde{V}_k(\psi_M), f^*T^0) \xrightarrow{G_*} \Omega_n(\tilde{V}_k(\psi_N),TN^0)\]

\[\Omega_n(M, f^*T^0) \xrightarrow{f_*} \Omega_n(N, TN^0)\]

\[\gamma'_M \xrightarrow{\gamma_N} \Omega_{n-k-1}(M \times P^\infty, \phi_M) \xrightarrow{F_*} \Omega_{n-k-1}(N \times P^\infty, \phi_N)\]

where the vertical sequences are obtained from (III), \(G_*\) and \(F_*\) are given in [S], \(\psi_M = \varepsilon^{n+k} - TM \oplus e^k\) and \(\psi_N = \varepsilon^{n+k} - TN \oplus e^k\).

We observe that \((\pi'_M)_*\) is the induced map of \(\pi_M\) in normal bordism groups with virtual bundle \(f^*TN^0\).

If \(N\) immerses in \(\mathbb{R}^{n+k}\), \((\pi_N)_*\) is sobjective and since \(f_* : H_i(M, \mathbb{Z}_2) \rightarrow H_i(N, \mathbb{Z}_2)\) is an isomorphism for \(i \geq 0\), then \(F_*\) is a \(C\)-monomorphism. Therefore \((\pi'_M)_*\) is a \(C\)-epimorphism and since every element of the image of \(\gamma'_M\) has order a potency of 2 ([S]), we conclude that \((\pi'_M)_*\) is an epimorphism.

Let us consider now the commutative diagram

\[
\begin{array}{ccc}
\pi_{n+p}^*(T\tilde{\alpha}) & \xrightarrow{\pi_{n+p}^*(Tf^*(\tilde{\beta}))} & \pi_{n+p}^*(Tf^*\alpha) \\
(\pi_M)_* & & (\pi'_M)_* \\
\pi_{n+p}^*(T\alpha) & \xrightarrow{\pi_{n+p}^*(Tf^*\alpha)} & \pi_{n+p}^*(Tf^*\alpha) \\
\end{array}
\]

where \(\tilde{\beta}\) and \(\tilde{\alpha}\) denote the pull back of \(\beta\) and \(\alpha\) by \(\pi_N\) and \(\pi_M\), respectively; then two horizontal maps are \(C\)-isomorphisms.

We conclude that \((\pi_M)_*\) is a \(C\)-epimorphism and since the elements of the image of \(\gamma_M\) has order a potency of 2, the result follows.

\[\square\]

4. APPLICATIONS

Let \(M\) and \(N\) be closed smooth manifolds of dimension \(n\) and \((n + k)\), respectively and let \(f : M \rightarrow N\) be a continuous map. Define \(U_f \in H^k(N, \mathbb{Z}_2)\) to be the image of the
fundamental class $[M] \in H_n(M, \mathbb{Z}_2)$ by the composite

$$H_n(M, \mathbb{Z}_2) \xrightarrow{f_*} H_n(N, \mathbb{Z}_2) \xrightarrow{D_N^{-1}} H^k(N, \mathbb{Z}_2)$$

where $D_N$ denotes the Poincaré duality isomorphism.

We also consider the following commutative diagram:

$$
\begin{array}{ccc}
H^p(N, \mathbb{Z}_2) & \xrightarrow{\cup U_f} & H^{p+k}(N, \mathbb{Z}_2) \\
\downarrow D_M \circ f^* & & \downarrow D_N \\
H_{n-p}(M, \mathbb{Z}_2) & \xrightarrow{f_*} & H_{n-p}(N, \mathbb{Z}_2)
\end{array}
$$

**Theorem 1:** Let $M$ and $N$ be closed smooth manifolds of dimension $n$. Suppose that

$$H_i(M, \mathbb{Z}_2) \simeq H_i(N, \mathbb{Z}_2), \text{ for all } i \geq 0$$

and there exists $f : M \to N$ with $\deg f = 1$. Then $f_* : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ is an isomorphism, for $i \geq 0$.

**Proof:** Since $M$ and $N$ have dimension $n$, we have that $U_f \in H^0(N, \mathbb{Z}_2)$ and $U_f = \deg_2 f$.

Therefore $\cup U_f$ is a multiple of $\deg_2 f$ and since $\deg_2 f = 1$, we have that $\cup U_f : H^p(N, \mathbb{Z}_2) \to H^p(N, \mathbb{Z}_2)$ is the identity map, for $p \geq 0$ and

$$f_* : H_{n-p}(M, \mathbb{Z}_2) \to H_{n-p}(N, \mathbb{Z}_2)$$

is onto, for all $p \geq 0$. But $H_i(M, \mathbb{Z}_2) \simeq H_i(N, \mathbb{Z}_2)$, $i \geq 0$, and the result follows. □

**Corollary 2:** Let $M$ and $N$ be closed smooth $n$-manifolds with isomorphic homology groups. Suppose that there exists $f : M \to N$ with $\deg_2 f = 1$. Then $M$ immerses in $\mathbb{R}^{n+k}$, $5 \leq n < 2k$, if and only if $N$ immerses. □

Let $M$ and $N$ be closed smooth $n$-manifolds. Given $x_0 \in M^n$ and $y_0 \in N^n$, let us take $D_1^n$ and $D_2^n$ disks containing $x_0$ and $y_0$, respectively, for which there exists a homeomorphism $h : D_1^n \to D_2^n$ with $h(x_0) = y_0$.

We denote by $A = \partial D_1$, $M_{n-1} = M^{(n-1)} \cup A$, where $M^{(n-1)}$ is the $(n-1)$-skeleton of $M$, $Y = N - h(D_1)$, $f_0 = h|_A$ and

$$\chi_{n-1}^n : H^n(M, A, \pi_{n-1}(Y)) \to H^n(M, A, H_{n-1}(Y))$$
is the homomorphism induced in cohomology by the Hurewicz homomorphism.

Let us suppose that \( f_0 \) extends to \( M_{n-1}, Y \) is \((n-1)\)-simple and \( H_{n-1}(A, \mathbb{Z}) \) is a free group.

**Theorem 3:** Suppose that \( M^n \) and \( N^n \) are such that \( H_*(M, \mathbb{Z}_2) \cong H_*(N, \mathbb{Z}_2) \).

If \( \chi_n^{n-1} \) is a monomorphism and there exists a homomorphism \( \psi : H_n(M, \mathbb{Z}) \to H_n(N, \mathbb{Z}) \) such that \( (f_0)_* = \psi \circ i_* \), with \( i_* : H_*(A, \mathbb{Z}) \to H_*(M, \mathbb{Z}) \) induced by the inclusion, then there exists \( f : M \to N \) with \( \deg_2 f = 1 \).

**Proof:** In these conditions \( f_0 \) extends to \( f : M \to N \) (see [B]) with \( f(M - D_1) = N - f(D_1) \).

By excision \( H_n(M, \mathbb{Z}_2) \) (resp. \( H_n(N, \mathbb{Z}_2) \)) is isomorphic to \( H_n(M, M - x_0, \mathbb{Z}_2) \) (resp. \( H_n(N, N - y_0, \mathbb{Z}_2) \)), which is isomorphic to \( H_n(D_1, D_1 - x_0, \mathbb{Z}_2) \) (resp. \( H_n(f(D_1), f(D_1) - y_0, \mathbb{Z}_2) \)) and the result follows.

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