Existence Results for Partial Neutral Integrodifferential Equations with Unbounded Delay.

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We prove the existence of a mild solution for a partial neutral functional integrodifferential equation with unbounded delay using the Leray Schauder Alternative.

1. INTRODUCTION

The purpose of this paper is to prove the existence of mild solutions for a class of partial neutral functional integrodifferential equations with unbounded delay described in the form

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t, \int_0^t h(t, s, x_s)ds), \quad t \in I = [\sigma, T],$$

$$x_\sigma = \varphi \in B,$$

(1)

where $A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(T(t))_{t \geq 0}$, on a Banach space $X$; the history $x_t: (-\infty, 0] \to X$, $x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space $B$ defined axiomatically; $0 \leq \sigma < T$ and $G: [\sigma, T] \times B \to X$, $F: [\sigma, T] \times B \times X \to X$ and $h: [\sigma, T] \times [\sigma, T] \times B \to X$ are appropriate functions.

Recently, Balachandran studied in [1] the existence of mild solutions for a neutral integrodifferential equation with finite delay modeled in the form

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t, \int_0^t h(t, s, x_s)ds), \quad t \in [0, T],$$

$$x_\sigma = \varphi \in C([-r, 0]: X),$$

where $A$ is the infinitesimal generator of a compact semigroup of bounded linear operators, $(T(t))_{t \geq 0}$, on $X$. In relation with the cited paper, it is necessary to make the following

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observation. The assumption (i), see pp. 95, "A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in $X$ such that

$$
\| T(t) \| \leq M_1, \quad \text{for some } M_1 \geq 1 \quad \text{and} \quad \| AT(t) \| \leq M_2, \quad M_2 > 0,
$$

implies that $A$ is bounded and that $X$ is a finite dimensional space. In fact, under this condition

$$
\| Ax \| = \lim_{t \to 0} \| T(t)Ax \| = \lim_{t \to 0} \| AT(t)x \| \leq M_2 \| x \|,
$$

for every $x \in D(A)$, thus $A$ is bounded on $D(A)$ and then bounded on $X$ since $A$ is a closed operator with dense domain in $X$. Since $A$ is bounded, the semigroup $(T(t))_{t \geq 0}$ is uniformly continuous and thus, $T(0) = I$ is compact.

It's clear from the previous remark that the neutral equation studied by Balachandran does not include partial integrodifferential equations with delay. For this reason, we discuss in this paper the solvability of the initial value problem (1) in the case where $A$ is a possibly unbounded infinitesimal generator of an analytic semigroup of bounded linear operators on $X$. The main result is proved using Leray Schauder Alternative and under the assumption that the sets $T(\epsilon)(-A)^3G([\sigma,T] \times B_r(\varphi,\mathcal{B}))$ and $T(\epsilon)F([\sigma,T] \times B_r(\varphi,\mathcal{B}) \times B_r(x,X))$, $\epsilon > 0$, $r > 0$, are relatively compact in $X$.

Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for neutral functional differential equations is the Hale book [4] and the references therein. The work in partial neutral functional differential equations with unbounded delay was initiated by Hernández & Henríquez in [7, 8]. In these papers, Hernández & Henríquez proved the existence of mild, strong and periodic solutions for a neutral equation modeled in the form

$$
\frac{d}{dt}(x(t) + G(t,x_t)) = Ax(t) + F(t,x_t), \quad t \geq \sigma,
$$

$$
x_\sigma = \varphi \in \mathcal{B},
$$

where $A$ is the infinitesimal generator of an analytic semigroup of linear operators on $X$ and $\mathcal{B}$ is a phase space defined axiomatically. In general, the results were obtained using the semigroup theory and the Sadovskii fixed point theorem (see [13]). Other related papers are Hernández [9] and Henríquez [5].

Throughout this paper, $X$ will be a Banach space provided with norm $\| \cdot \|$ and $A : D(A) \rightarrow X$ will be the infinitesimal generator of an analytic semigroup, $(T(t))_{t \geq 0}$, of bounded linear operators on $X$. For the theory of strongly continuous semigroup, refer to Pazy [12]. We will point out here some notations and properties that will be used in this work. It is well known that there exist $\tilde{M} \geq 1$ and $w \in \mathbb{R}$ such that $\| T(t) \| \leq \tilde{M} e^{wt}$ for every $t \geq 0$. If $T$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in $X$ and the expression $\| x \|_\alpha = \| (-A)^\alpha x \|$ defines a norm in $D(-A)^\alpha$. If $X_\alpha$ represents the space...
$D(-A)\alpha$ endowed with the norm $\| \cdot \|_\alpha$ then the following properties are well known ([12], pp. 74):

**Lemma 1.1.** If the previous conditions hold:

1) Let $0 < \alpha \leq 1$. Then $X_\alpha$ is a Banach space.
2) If $0 < \beta \leq \alpha$ then $X_\alpha \to X_\beta$ is continuous.
3) For every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that

$$\| (A)^\alpha T(t) \| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq T.$$  

In this work we will employ an axiomatic definition of the phase space $B$ introduced by Hale and Kato [3]. In order to establish the axioms of the space $B$ we follow the terminology used in Hino-Murakami-Naito [10], and thus, $B$ will be a linear space of functions mapping $(-\infty, 0]$ into $X$, endowed with a seminorm $\| \cdot \|_B$. We will assume that $B$ satisfies the following axioms:

**A** If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$, then for every $t \in [\sigma, \sigma + a)$, the following conditions hold:

i) $x_t$ is in $B$.

ii) $\| x(t) \| \leq H \| x_t \|_B$.

iii) $\| x_t \|_B \leq K(t - \sigma) \sup \{ \| x(s) \| : \sigma \leq s \leq t \} + M(t - \sigma) \| x_\sigma \|_B$.

where $H > 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, $K$ is continuous, $M$ is locally bounded and $H, K, M$ are independent of $x(\cdot)$.

**A1** For the function $x(\cdot)$ in (A), $x_t$ is a $B$-valued continuous function on $[\sigma, \sigma + a)$.

**B** The space $B$ is complete.

**Example 1.1.** We consider now the phase space $B := C_r \times L^p(g; X)$, $r \geq 0$, $1 \leq p < \infty$ in [10], which consists of all classes of functions $\varphi : (\infty, 0) \to X$ such that $\varphi$ is continuous on $[-r, 0]$, Lebesgue-measurable and $g | \varphi(\cdot) |^p$ is Lebesgue integrable on $(-\infty, -r)$, where $g : (-\infty, -r) \to \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\| \cdot \|_B$ is defined by

$$\| \varphi \|_B := \sup \{ \| \varphi(t) \| : -r \leq t \leq 0 \} + \left( \int_{-\infty}^{-r} g(\theta) \| \varphi(\theta) \|^p d\theta \right)^{1/p}.$$  

We will assume that $g$ satisfies conditions (g-6) and (g-7) in the terminology of [10]. This means that $g$ is integrable on $(-\infty, -r)$ and that there exists a non-negative and locally bounded function $\gamma$ on $(-\infty, 0]$ such that

$$g(\xi + \theta) \leq \gamma(\xi) g(\theta),$$  

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for all \( \xi \leq 0 \) and \( \theta \in (-\infty, -r) \setminus \mathcal{N}_\xi \), where \( \mathcal{N}_\xi \subseteq (-\infty, -r) \) is a set with Lebesgue measure zero. In this case, \( \mathcal{B} \) is a phase space which verifies axioms (A), (A-1) and (B) ([10], Theorem 1.3.8).

The paper is organized as follows. In section 2, we discuss the existence of mild solutions for the initial value problem (1). In section 3, an example is considered. Our results are based on the properties of analytic semigroups and the ideas and techniques in Hernández [7, 8], Henríquez [6] and Balachandran [1].

Finally, we remark that for the proof of our existence theorem we shall use of the following result which is referred to as the Leray Schauder Alternative.

**Theorem 1.1.** [12] (Leray Schauder Alternative) Let \( D \) be a convex subset of a Banach space \( X \) and assume that \( 0 \in S \). Let \( F : D \to D \) be a completely continuous map. Then \( F \) has a fixed point in \( D \) or the set \( \{x \in D : x = \lambda F(x), \ 0 < \lambda < 1\} \) is unbounded.

## 2. EXISTENCE RESULTS

In this section, we study the existence of mild solutions for the abstract Cauchy problem (1). Henceforth, we will assume that \( A \) is the infinitesimal generator of an analytic semigroup, \( (T(t))_{t \geq 0} \), of bounded linear operators on \( X \). Further, to avoid unnecessary notations, we suppose that \( 0 \in \rho(A) \) and that for \( 0 < \vartheta < 1 \), see Lemma 1.1,

\[
\| T(t) \| \leq \bar{M}, \quad t \geq 0 \quad \text{and} \quad \| (-A)^\vartheta T(t) \| \leq \frac{C_\vartheta}{\vartheta}, \quad t \in (0, T],
\]

for some constants \( \bar{M}, C_\vartheta \).

**Remark 2.1.** In the rest of this work, to simplify notations, we only consider the case \( \sigma = 0 \).

**Definition 2.1.** We will say that a function \( x : (-\infty, T] \to X \) is a mild solution of the abstract Cauchy problem (1) if: \( x_0 = \varphi \); the restriction of \( x(\cdot) \) to the interval \( I = [0, T] \) is continuous; for each \( 0 \leq t < T \), the function \( AT(t - s)G(s, x_s), s \in [0, t) \), is integrable and

\[
x(t) = T(t)(\varphi(0) + G(0, \varphi)) - G(t, x_t) - \int_0^t AT(t - s)G(s, x_s)ds
+ \int_0^t T(t - s)F(s, x_s) \int_s^t h(s, \tau, x_\tau) d\tau ds, \quad t \in I = [0, T].
\]

Along this paper we assume that the following assumptions hold.
Assumptions:

**H**<sub>1</sub> The function \( F : I \times \mathcal{B} \times X \rightarrow X \) satisfies the following conditions:

(i) For every \( t \in I \), the function \( F(t, \cdot) : \mathcal{B} \times X \rightarrow X \) is continuous.

(ii) For each \((\psi, x) \in \mathcal{B} \times X\), the function \( F(\cdot, \psi, x) : I \rightarrow X \) is strongly measurable.

**H**<sub>2</sub> There exist constants \( 0 < \beta < 1, c_1, c_2 \) such that the function \( G \) is \( X_{\beta} \)-valued, \((-A)^{\beta}G(\cdot)\) is continuous and

\[
\|(-A)^{\beta}G(t, \psi)\| \leq c_1 \|\psi\|_B + c_2, \quad (t, \psi) \in [0, T] \times \mathcal{B}.
\]

**H**<sub>3</sub> The function \( h : I \times I \times \mathcal{B} \rightarrow X \) satisfies the following conditions:

(i) For every \((t, s) \in I \times I\), the function \( h(t, s, \cdot) : \mathcal{B} \rightarrow X \) is continuous.

(ii) For each \( \psi \in \mathcal{B}\), the function \( h(\cdot, \psi) : I \times I \rightarrow X \) is strongly measurable.

**H**<sub>4</sub> There exist continuous functions \( m_F, m_h : I \rightarrow [0, \infty) \) such that

\[
\|F(t, \psi, x)\| \leq m_F(t)\Omega_F(\|\psi\|_B + \|x\|),
\]

\[
h(t, s, \psi) \| \leq m_h(s)\Omega_h(\|\psi\|_B),
\]

for every \( \psi \in \mathcal{B}, x \in X \) and \( 0 \leq s \leq t \leq T \); where \( \Omega_F, \Omega_h : [0, \infty) \rightarrow (0, \infty) \) are continuous nondecreasing functions.

\( G_\varphi \) Let \( \varphi \in \mathcal{B}; y : (-\infty, T] \rightarrow X \) be the function defined by

\[
y(t) = \begin{cases} T(t)\varphi(0) & \text{for } t \geq 0, \\ \varphi(t) & \text{for } t \leq 0, \end{cases}
\]

and \( S(T) \) be the space \( S(T) = \{x : (-\infty, T] \rightarrow X : x_0 = 0; x \in C([0, T] : X)\} \) endowed with the norm of uniform convergence on \([0, T]\). We said that the initial value problem (1) verifies the condition \( G_\varphi \) if for every bounded \( Q \subset S(T) \), the set of functions \( \{t \rightarrow G(t, x_t + y_t) : x \in Q\} \) is equicontinuous on \([0, T]\).

The proof of the next result follows using the steps in the proof of Lemma 5.6.7 in Pazy [12] and the Gronwall’s inequality. In the next Lemma, \( \Gamma(\cdot) \) is the gamma function.

**Lemma 2.1.** Let \( v(\cdot), w(\cdot) : [0, T] \rightarrow [0, \infty) \) be continuous functions. If \( w(\cdot) \) is nondecreasing and there are constants \( \vartheta > 0 \) and \( 0 < \alpha < 1 \) such that

\[
v(t) \leq w(t) + \vartheta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}}ds, \quad t \in [0, T],
\]

then

\[
v(t) \leq e^{-\frac{\vartheta t}{\Gamma(\alpha)}} \sum_{j=0}^{n-1} \left( \frac{\vartheta T^\alpha}{\alpha} \right)^j \cdot w(t)
\]
For every $t \in [0, T]$ and every $n \in \mathbb{N}$ such that $n\alpha > 1$.

**Notation 2.1.** For a bounded function $\xi : [0, T] \to [0, \infty)$ and $0 \leq t \leq T$ we define the number $\xi_t$ by $\xi_t = \sup \{ \xi(\theta) : \theta \in [0, t] \}$. In particular, this enables us to rewrite the inequality $iii)$ in axiom (A) in the form $\| x_t \| \leq M_T \| \varphi \|_B + K_T \| x \|_t$. In addition, $B_r[x : Z]$ will denote the closed ball in a metric space $Z$ with center at $x$ and radius $r$.

**Theorem 2.1.** Let $\varphi \in B$ and assume that assumptions $H_1 - H_4$ and condition $G_\varphi$ hold. Suppose, furthermore, that the following conditions are fulfilled:

(a) For every $r > 0$ and every $\epsilon > 0$, there are compact sets $W_{c, r} \subset X$ such that $T(\epsilon)(-A)^{\beta}G(s, \psi) \in W_{c, r}^1$ and $T(\epsilon)F(s, \psi, x) \in W_{c, r}^2$ for every $s \in [0, T]$, $\psi \in B_r[0, B]$ and $x \in B_r[0, X]$.

(b) $\mu = \| ( -A )^{-\beta} \| K_Tc_1 < 1$ and

$$
\int_0^T \frac{bK_T \hat{M}}{1 - \mu} m_F(s) + m_h(s)) ds < \int_{\infty}^0 \frac{ds}{\Omega_F(s) + \Omega_h(s)},
$$

where $C = ab$,

$$
b = \frac{\epsilon^{n\gamma}T(\epsilon)\gamma^{\gamma \beta}}{n\gamma(\gamma + 1)} \sum_{j=0}^{n-1} \frac{b_j T^j}{\beta}; \quad b_1 = \frac{K_T c_1 - \beta c_1}{1 - \mu}; \quad a = \frac{M_T \| \varphi \|_B + K_T a_1}{1 - \mu};
$$

$$
a_1 = \| \varphi \|_B (\hat{M}H + c_1 \| ( -A )^{-\beta} \| ( \hat{M} + M_T ) + \| ( -A )^{-\beta} \| c_2 (\hat{M} + 1) + c_1 - \beta c_2 T^\beta }{\beta},
$$

$C_1 - \beta$ is the constant in (2) and $n$ is the first natural number such that $n\beta > 1$.

Then there exists a mild solution of the initial value problem (1).

**Proof:** In order to use the Leray Schauder Alternative, we obtain a priori estimates for the solutions of the integral equation

$$
x(t) = \lambda T(t)(\varphi(0) + G(0, \lambda \varphi)) - \lambda G(t, x_t) - \lambda \int_0^t AT(t - s)G(s, x_s) ds
$$

$$
+ \lambda \int_0^t T(t - s)F(s, x_s, \int_0^s h(s, \tau, x_{\tau}) d\tau) ds, \quad t \in [0, T],
$$

where $x_0 = \lambda \varphi$. Let $\lambda \in (0, 1)$ and $x(\cdot)$ be a solution of (6). Using the assumptions $H_1 - H_4$, Lemma 1.1, remark 2.1 and the notation

$$
\| x_t \|_B \leq M_T \| \varphi \|_B + K_T \| x \|_t := v(t)
$$
we find that
\[
\|x(t)\| \leq \hat{M}(H \| \varphi \|_{\mathcal{B}} + \| (\mathcal{A})^{-\beta} \| (c_1 \| \varphi \|_{\mathcal{B}} + c_2) + \| (\mathcal{A})^{-\beta} \| (c_1 \| x_t \|_{\mathcal{B}} + c_2)
+ \int_0^t \| (\mathcal{A})^{1-\beta} T(t-s) \||\mathcal{A}^{\beta} G(s, x_s) \| \, ds
+ \hat{M} \int_0^t m_F(s) \Omega_F \left(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) \, d\tau\right) \, ds
\]
\[
\leq \hat{M}(H \| \varphi \|_{\mathcal{B}} + \| (\mathcal{A})^{-\beta} \| (c_1 \| \varphi \|_{\mathcal{B}} + c_2) + \| (\mathcal{A})^{-\beta} \| c_1 v(t) + \| (\mathcal{A})^{-\beta} \| c_2 + \int_0^t \| (\mathcal{A})^{1-\beta} \| x_s \|_{\mathcal{B}} \, ds + \frac{C_{1-\beta} c_1}{(t-s)^{1-\beta}}
+ \hat{M} \int_0^t m_F(s) \Omega_F \left(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) \, d\tau\right) \, ds
\]
and since \(v(\cdot)\) is nondecreasing,
\[
\|x(t)\| \leq a_1 + \| (\mathcal{A})^{-\beta} \| c_1 v(t) + C_{1-\beta} c_1 \int_0^t \frac{v(s)}{(t-s)^{1-\beta}} \, ds
+ \hat{M} \int_0^t m_F(s) \Omega_F \left(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) \, d\tau\right) \, ds,
\]
where \(a_1 = \hat{M}(H + c_1 \| (\mathcal{A})^{-\beta} \| \| \varphi \|_{\mathcal{B}} + \| (\mathcal{A})^{-\beta} \| c_2) + c_2(\| (\mathcal{A})^{-\beta} \| + \frac{C_{1-\beta} c_1}{\beta})\).

Employing the last inequality and the definition of \(v(\cdot)\) we get
\[
v(t) \leq M_T \| \varphi \|_{\mathcal{B}} + K_T a_1 + K_T \| (\mathcal{A})^{-\beta} \| c_1 v(t)
+ K_T C_{1-\beta} c_1 \int_0^t \frac{v(s)}{(t-s)^{1-\beta}} \, ds
+ K_T \hat{M} \int_0^t m_F(s) \Omega_F \left(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) \, d\tau\right) \, ds
\]
and hence
\[
v(t) \leq \frac{M_T \| \varphi \|_{\mathcal{B}} + K_T a_1}{1 - \mu} + \frac{K_T C_{1-\beta} c_1}{1 - \mu} \int_0^t \frac{v(s)}{(t-s)^{1-\beta}} \, ds
+ \frac{K_T \hat{M}}{1 - \mu} \int_0^t m_F(s) \Omega_F \left(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) \, d\tau\right) \, ds,
\]
since \(\mu = \| (\mathcal{A})^{-\beta} \| K_{\mathcal{T}} c_1 < 1\). In this condition, Lemma 2.1 implies that
\[
v(t) \leq b \left( a + \frac{K_T \hat{M}}{1 - \mu} \int_0^t m_F(s) \Omega_F \left(v(s) + \int_0^s m_h(\tau) \Omega_h(v(\tau)) \, d\tau\right) \, ds \right) := w(t)
\]

\[
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\]
where \( a = \frac{M_T \| \varphi \|_n + K \tau_x}{1 - \mu} \), \( b = e^{\frac{\beta \gamma_1 (n-1)}{1 - \mu}} \cdot \sum_{j=0}^{n-1} \left( \frac{\beta \gamma_1}{\mu} \right)^j \), \( b_1 = \frac{K \tau C_{\lambda - \beta x}}{1 - \mu} \) and \( n \) is the first natural number such that \( n\beta > 1 \). From the definition of \( w(\cdot) \), we see that

\[
\dot{w}(t) \leq \frac{bK_T \tilde{M}}{1 - \mu} m_F(t) \Omega_F \left( w(t) + \int_0^t m_h(\tau) \Omega_h(w(\tau)) d\tau \right)
\]

(8)

since \( \Omega_F(\cdot) \) and \( \Omega_h(\cdot) \) are nondecreasing. Let \( z(t) = w(t) + \int_0^t m_h(\tau) \Omega_h(w(\tau)) d\tau \). Then, \( z(0) = w(0) = C \) and from (8)

\[
\dot{z}(t) = \dot{w}(t) + m_h(t) \Omega_h(w(t)) \leq \frac{bK_T \tilde{M}}{1 - \mu} m_F(t) \Omega_F(z(t)) + m_h(t) \Omega_h(z(t))
\]

\[
\leq \left( \frac{bK_T \tilde{M}}{1 - \mu} m_F(t) + m_h(t) \right) (\Omega_F(z(t)) + \Omega_h(z(t))).
\]

Consequently, from condition (b),

\[
\int_{z(0)}^{z(t)} \frac{ds}{\Omega_F(s) + \Omega_h(s)} \leq \int_0^t \left( \frac{bK_T \tilde{M}}{1 - \mu} m_F(s) + m_h(s) \right) ds < \int_C^1 \frac{1}{\Omega_F(s) + \Omega_h(s)} ds.
\]

This inequality permits to conclude that there exists \( N > 0 \) independent of \( x(\cdot) \) and \( \lambda \in (0, 1) \) such that \( \| x \|_t \leq z(t) \leq N \) for every \( t \in [0, T] \). Thus, the set

\[
\{ x \in C([0, T] : X) : x(\cdot) \text{ is solution of (6), } \lambda \in (0, 1) \}
\]

is bounded in \( C([0, T] : X) \).

Next we rewrite equation (3) as follows. Let \( y : (-\infty, T] \to X \) be the function in (5). If \( u(\cdot) \) is a mild solution of (1), we can decompose it as \( u(t) = x(t) + y(t) \), which implies that \( u_t = x_t + y_t \), \( x_0 = 0 \) and that \( x(\cdot) \) is a solution of the integral equation

\[
x(t) = T(t)G(0, \varphi) - G(t, x_t + y_t) - \int_0^t AT(t-s)G(s, x_s + y_s)ds
\]

\[
+ \int_0^t T(t-s)F(s, x_s + y_s) \int_0^s h(s, \tau, x_\tau + y_\tau) d\tau ds,
\]

moreover, using that \( A \) is closed, the relation \( T(t)x - x = A \int_0^t T(s)x ds, x \in X \), and that \( (-A)^\beta G(\cdot) \) is continuous, follows that

\[
x(t) = T(t)(G(0, \varphi) - G(t, x_t + y_t)) - \int_0^t AT(t-s)G(s, x_s + y_s)ds
\]

\[
+ \int_0^t AT(t-s)G(t, x_t + y_t)ds + \int_0^t T(t-s)F(s, x_s + y_s) \int_0^s h(s, \tau, x_\tau + y_\tau) d\tau ds.
\]
On the space \( S(T) = \{ x : (-\infty, 0] \rightarrow X : x_0 = 0, \ x \in C([0, T] : X) \} \) provided with the uniform convergence topology, we define the map \( \Gamma : S(T) \rightarrow S(T) \) by

\[
\Gamma x(t) = \begin{cases} 
0, & t \leq 0, \\
T(t)(G(0, \varphi) - G(t, x_t + y_t)) - \int_0^t AT(t - s)G(s, x_s + y_s)ds \\
+ \int_0^t AT(t - s)G(t, x_t + y_t)ds \\
+ \int_0^t T(t - s)F(s, x_s + y_s, \int_0^s h(s, \tau, x_\tau + y_\tau) d\tau)ds, & t \in [0, T].
\end{cases}
\]

These expressions leads us to define the maps \( \Gamma_i : S(T) \rightarrow S(T), \ i = 1, 2, 3, 4, \) by

\[
\Gamma_i x(t) = \begin{cases} 
0, & i = 1, 2, 3, 4, \\
T(t)(G(0, \varphi) - G(t, x_t + y_t)), \\
- \int_0^t AT(t - s)G(s, x_s + y_s)ds, \\
\int_0^t AT(t - s)G(t, x_t + y_t)ds, \\
\int_0^t T(t - s)F(s, x_s + y_s, \int_0^s h(s, \tau, x_\tau + y_\tau) d\tau)ds,
\end{cases}
\]

for \( 0 \leq t \leq T. \)

Next we will prove that the functions \( \Gamma_i \) are well defined and with values in \( S(T). \) Moreover, each \( \Gamma_i \) is completely continuous.

It is clear from \( H_1, H_3 \) and \( H_4 \) that \( \Gamma_1 \) and \( \Gamma_4 \) are well defined maps with values in \( S(T). \) Moreover, since \( G \) is \( X_\beta \)-valued and \( (-A)^\beta G \) is continuous, then both \( s \rightarrow (-A)^\beta G(s, x_s + y_s) \) and \( s \rightarrow G(s, x_s + y_s) \) are continuous. In addition, considering that \( (T(t))_{t \geq 0} \) is an analytic semigroup (see [12]), the operator function \( s \rightarrow AT(t - s) \) is continuous in the uniform operator topology on \( [0, t] \) and thus, \( s \rightarrow AT(t - s)G(s, x_s + y_s) \) is continuous on \( [0, t]. \) Applying the inequality (2), we obtain that

\[
\| (-A)T(t - s)G(s, x_s + y_s) \| = \| (-A)^{1-\beta}T(t - s)(-A)^\beta G(s, x_s + y_s) \| \\
\leq \frac{C_{1-\beta} \text{Cte}}{(t - s)^{1-\beta}} \tag{9}
\]

which, by Bochner’s Theorem (see [11]), implies that \( \| AT(t - s)G(s, x_s + y_s) \| \) is integrable on \( [0, t]. \) This concludes the proof that \( \Gamma_2 \) is also a well defined map with values
in $S(T)$. The same conclusion holds for $\Gamma_2$. Thus, $\Gamma$ is a well defined map with values in $S(T)$.

On the other hand, from axiom (A), the map $[0,T] \times S(T) \to B; (s,x) \to x_s$ is continuous. If $(x^n)_{n\in\mathbb{N}}$ is a convergent sequence in $S(T)$ with limit $x$, then the set $W = \{(s,x_s), (s,x^n_s) : s \in [0,T], n \in \mathbb{N}\}$ is compact in $[0,T] \times B$ and hence $(-A)^3G(t)\| \cdot \|$ is uniformly continuous on $W$. Since $x^n$ converges to $x$ in $S(T)$, it follows that $(-A)G(s,x^n_s) \to (-A)G(s,x_s)$ uniformly on $[0,T]$, which jointly with the estimate (8) permits to conclude that $\Gamma_2(x^n_t) \to \Gamma_2(x)$. Using similar arguments and the assumptions $H_1$, $H_3$ and $H_4$ we can prove that $\Gamma_1, \Gamma_3, \Gamma_4$ are continuous, we omit details. Thus, $\Gamma$ is continuous.

Next, we will prove that each $\Gamma_i$ is completely continuous. By Ascoli’s theorem, it is sufficient to show that the set $\Gamma_i(B_r[0,S(T)])$ is equicontinuous on $[0,T]$ and that $\Gamma_i(B_r[0,S(T)])(t) = \{ \Gamma_i x(t) : x \in B_r[0,S(T)] \}$ is relatively compact in $X$ for every $t \in [0,T]$.

In what follow, $r > 0$ is fixed and $B_r = B_r[0,S(T)]$. Now, we will study the property for each $\Gamma_i$. At first, we observe that

$$\| x_t + y_t \|_B \leq K_T r + (M_T + K_T M_H) \| \varphi \|_B := r^*.$$  
(10)

for every $x \in B_r[0,S(T)]$ and every $t \in [0,T]$.

$\Gamma_1$. Firstly we study the equicontinuity of $\Gamma_1 B_r$. Let $0 < \epsilon < t_0 < t < T$ and $W^{1,r*}_{r,\epsilon}$ be the compact set in (a). Using that the set of functions $\{ T(s)x : x \in W^{1,r}_{\epsilon,r} \}$ is equicontinuous on $[0,T]$ and condition $G_x$ we choose $0 < \delta < \epsilon$ such that

$$\| G(t,x_t + y_t) - G(t_0,x_{t_0} + y_{t_0}) \| < \epsilon, \quad x \in B_r,$$

$$\| T(t + \delta)x - T(t)x \| < \epsilon, \quad x \in W^{1,r}_{\epsilon,r*},$$

when $| t - t_0 | < \delta$. Consequently

$$\| \Gamma_1 x(t_0) - \Gamma_1 x(t) \| \leq \| (T(t_0) - T(t))G(0,\varphi) \| + \| T(t_0) \| \| G(t_0,x_{t_0} + y_{t_0}) - G(t,x_t + y_t) \| + \| (\epsilon - A^{-\beta}) T(t_0 - \epsilon) - T(t_0 - \epsilon) T(\epsilon) (\epsilon - A^{-\beta}) G(t,x_t + y_t) \|$$

and hence,

$$\| \Gamma_1 x(t_0) - \Gamma_1 x(t) \| \leq \| (T(t_0) - T(t))G(0,\varphi) \| + \| T(t_0) \| + \| (\epsilon - A^{-\beta}) \| \epsilon$$

when $| t - t_0 | < \delta$, which implies that $\Gamma_1(B_r)$ is equicontinuous from the right side at $t_0$.

The equicontinuity in $t_0 \geq 0$ is proved in similar form, we omit details.

It’s clear that $\Gamma_1(B_r)(0)$ is compact, moreover, for $t > 0$ and $0 < \epsilon < t$

$$\Gamma_1(B_r)(t) \subset \{ T(t)G(0,\varphi) + T(t - \epsilon)(\epsilon - A^{-\beta})T(\epsilon)(\epsilon - A^{-\beta})G(t,x_t + y_t) : x \in B_r \}$$

$$\subset T(t)G(0,\varphi) + T(t - \epsilon)(\epsilon - A^{-\beta})W^{1,r}_{\epsilon,r*},$$
where $W^1_{x_1}$ is the compact set in (a). Since $(-A)^{-\beta}$ is continuous, it follows that $\Gamma_1(B_r)(t)$ is relatively compact. From these remarks, we infer that $\Gamma_1$ is a compact operator.

$\Gamma_2$ At first, we prove that $\Gamma_2 B_r$ is equicontinuous. Let $0 < t_0 < t \leq T$ and $0 < \epsilon < t_0$. For $x \in B_r$, we have that

$$
\left\| \Gamma_2 x(t) - \Gamma_2 x(t_0) \right\|
\leq \left\| \int_{t_0}^{t} (-A)^{1-\beta} (T(t - s) - T(t_0 - s)) (-A)^{\beta} G(s, x_s + y_s) ds \right\|
\leq \left\| \int_{t_0}^{t} (-A)^{1-\beta} T(t - s) - T(t_0 - s) (-A)^{\beta} G(s, x_s + y_s) ds \right\|
\leq \int_{t_0}^{t} (-A)^{1-\beta} T(t - s) (-A)^{\beta} G(s, x_s + y_s) ds
$$

then

$$
\left\| \Gamma_2 x(t) - \Gamma_2 x(t_0) \right\|
\leq \left\| (-A)^{1-\beta} [T(t - s - \epsilon) - T(t_0 - s - \epsilon)] T(\epsilon/2) (-A)^{\beta} G(s, x_s + y_s) \right\|
\leq 2MC_1 - \beta (c_1 r^* + c_2) (\epsilon^\beta/\beta) + (t - t_0)^\beta
\tag{11}
$$

where (10) has been used. Let $W^1_{x_1}$ be the compact set in (a). Since $T(\cdot)$ is analytic, the function $s \rightarrow AT(s)$ is continuous in the uniform operator topology on $(0, T)$, which implies that the set of functions $\{ s \rightarrow (-A)^{1-\beta} T(s)x : x \in W^1_{x_1} \}$ is equicontinuous on $[\frac{\epsilon}{2}, T]$. Let $0 < \delta < \epsilon$ such that

$$
\left\| (-A)^{1-\beta} T(s)x - (-A)^{1-\beta} T(s')x \right\| < \epsilon,
\quad x \in W^1_{x_1},
\tag{12}
$$

when $\frac{\epsilon}{2} \leq s, s' \leq T$ and $|s - s'| < \delta$. Thus, if $|t - t_0| < \delta < \epsilon$, substituting (12) into (11), we obtain that

$$
\left\| \Gamma_2 x(t) - \Gamma_2 x(t_0) \right\| \leq (t_0 - \epsilon) \epsilon + 2MC_1 - \beta (c_1 r^* + c_2) \epsilon^\beta/\beta
$$

which shows that $\Gamma_2 (B_r)$ is equicontinuous from the right side at $t_0$. In the same way, we can to prove that $\Gamma_2 (B_r)$ is equicontinuous at any $t_0 \geq 0$.

Now we will prove that $\Gamma_2(B_r)(t)$ is relatively compact for every $t \in [0, T]$. The case $t = 0$ is trivial. Let $0 < \epsilon < t \leq T$ and $W^1_{x_1}$ be the compact set in (a). For $x \in B_r$, we find that

$$
\Gamma_2 x(t) = \int_{t_0}^{t - \epsilon} (-A)^{1-\beta} T(t - s - \epsilon/2) T(\epsilon/2) (-A)^{\beta} G(s, x_s + y_s) ds
\begin{align*}
+ \int_{t - \epsilon}^{t} (-A)^{1-\beta} T(t - s) (-A)^{\beta} G(s, x_s + y_s) ds.
\end{align*}
\tag{13}
$$
Since $s \to (-A)^{\beta}T(s)$ is continuous on $[\frac{s}{2}, T]$, the set $B_{\epsilon} = \{(-A)^{1-\beta}T(s)x : s \in [\frac{s}{2}, T], x \in W^1_{\frac{s}{2}, r^*}\}$ is relatively compact in $X$. By the mean value theorem for the Bochner integral, see [11], and (13) we infer that

$$\Gamma_2 x(t) \in (t-\epsilon)\overline{\text{Conv}}(B_{\epsilon}) + C_{\epsilon}$$

where $\overline{\text{Conv}}(B_{\epsilon})$ is the convex hull of $B_{\epsilon}$ and $\text{Diam}(C_{\epsilon}) \leq 2C_{1-\beta}(c_1 r^* + c_2)\frac{\epsilon^2}{T^2}$. Using that $\overline{\text{Conv}}(B_{\epsilon})$ is relatively compact, we conclude that $\Gamma_2(B_{\epsilon})(t)$ is completely bounded and, therefore, relatively compact in $X$. This completes the proof that $\Gamma_2$ is completely continuous.

**$\Gamma_3$** The assertion can be proved arguing as in the previous case.

**$\Gamma_4$** Let $0 < t_0 < t \leq T$ and $0 < \epsilon < t_0$. If $x \in B_{\epsilon}$, we get

$$\| \Gamma_4 x(t) - \Gamma_4 x(t_0) \|$$

$$\leq \int_{t_0}^{t_0+\epsilon} \| (T(t-s) - T(t_0-s))F(s, x, y, \int_0^s h(s, \tau, x_\tau + y_\tau)\,d\tau) \| \, ds$$

$$+ \int_{t_0}^{t_0+\epsilon} \| [T(t-s) - T(t_0-s)]F(s, x, y, \int_0^s h(s, \tau, x_\tau + y_\tau)\,d\tau) \| \, ds$$

$$+ \int_{t_0}^{t_0+\epsilon} \| T(t-s)F(s, x, y, \int_0^s h(s, \tau, x_\tau + y_\tau)\,d\tau) \| \, ds = I_1 + I_2 + I_3. \quad (15)$$

We estimate each of the three terms separately. Let $r^{**} := \max\{r^*, \Omega_k(r^*) \int_0^T m_k(\tau)\,d\tau\}$ and $W^2_{\epsilon, r^{**}}$ be the compact set in (a). Since the functions $T(\cdot)x$, $x \in W^2_{\epsilon, r^{**}}$, are equicontinuous on $[0, T]$, there exists $0 < \delta < \epsilon$ such that

$$\| T(s)x - T(s')x \| < \epsilon, \quad x \in W^2_{\epsilon, r^{**}}, \quad (16)$$

if $|s - s'| < \delta$ and $0 \leq s, s' \leq T$. In this condition, for $x \in B_{\epsilon}$ we have that

$$T(\epsilon)F(s, x, y, \int_0^s h(s, \tau, x_\tau + y_\tau)\,d\tau) \in W^2_{\epsilon, r^{**}}$$

for every $s \in [0, T]$ and from (16),

$$\int_{t_0}^{t_0+\epsilon} \| (T(t-s) - T(t_0-s-\epsilon))T(\epsilon)F(s, x, y, \int_0^s h(s, \tau, x_\tau + y_\tau)\,d\tau) \| \, ds$$

$$\leq (t_0 - \epsilon)\epsilon.$$

Thus

$$I_1 \leq (t_0 - \epsilon)\epsilon, \quad (17)$$
for every \( x \in B_r \) and \( |t - t_0| < \delta \).

On the other hand

\[
I_2 \leq 2\tilde{M} \int_{t_0 - \epsilon}^{t_0} m_F(s) \Omega_F \left( \|x_s + y_s\|_B + \int_0^s m_h(\tau) \Omega_h(\|x_\tau + y_\tau\|_B) d\tau \right) ds,
\]

then

\[
I_2 \leq 2\tilde{M}\Omega_F(2r^{**}) \cdot \int_{t_0 - \epsilon}^{t_0} m_F(s) ds. \tag{18}
\]

Similarly

\[
I_3 \leq \tilde{M}\Omega_F(2r^{**}) \cdot \int_{t_0}^t m_F(s) ds. \tag{19}
\]

From inequalities (17), (18), (19) and (15) we infer that \( \Gamma_4(B_r) \) is equicontinuous from the right side at \( t_0 > 0 \). The equicontinuity in any \( t_0 \geq 0 \) is proved using the same arguments.

Now we will show that \( \Gamma_4(B_r)(t) \) is relatively compact for every \( t \in [0, T] \). The case \( t = 0 \) is trivial. Let \( 0 < \epsilon < t \leq T \). If \( r^{**} := \max\{r^\ast, \Omega_h(r^\ast) \cdot \int_0^T m_h(\tau) d\tau \} \) and \( W_{r^{**}}^2 \) is the compact set in (a), then the set \( B_\epsilon = \{ T(s)x : s \in [0, T], x \in W_{r^{**}}^2 \} \) is precompact in \( X \) and from the mean value theorem for the Bochner integral, it follows that

\[
\Gamma_4 x(t) = \int_0^{t-\epsilon} T(t-s-\epsilon)T(\epsilon)F(s,x_s+y_s,\int_0^s h(s,\tau,x_\tau+y_\tau) d\tau) ds
\]

\[
+ \int_{t-\epsilon}^t T(t-s)F(s,x_s+y_s,\int_0^s h(s,\tau,x_\tau+y_\tau) d\tau) ds
\]

\[
\in (t-\epsilon)\text{Conv}(B_\epsilon) + C_\epsilon
\]

where \( \text{Conv}(B_\epsilon) \) is the convex hull of \( B_\epsilon \) and

\[
\text{Diam}(C_\epsilon) \leq 2\tilde{M}\Omega_F(r^{**}) \int_{t-\epsilon}^t m_F(s) ds.
\]

This means that \( \Gamma_4 B_\epsilon(t) \) relatively compact in \( X \). Thus, \( \Gamma_4 \) is completely continuous.

These remarks enable us to assert that \( \Gamma = \sum_{i=1}^4 \Gamma_i \) is a completely continuous map on \( S(T) \). Finally, the Leray Schauder Alternative asserts that \( \Gamma \) has a fixed point which is a mild solution of the integrodifferential problem (1). The proof is complete.

**Corollary 2.1.** Let \( \varphi \in B \) and assume that \( H_1 - H_4, G_\varphi \) and condition (b) in Theorem 2.1 hold. If any of the following conditions hold:

\textbf{(a.1)} The functions \( F, (-A)^\beta G \) are completely continuous, or

\textbf{(a.2)} The semigroup \( (T(t))_{t \geq 0} \) is compact,
The function $\tilde{G} : S(T) \to C([0, T] : X)$; $x \to \tilde{G}(x)(t) = G(t, x, y_0)$ is completely continuous and for every $r > 0$ and every $\epsilon > 0$, there is a compact set $W^2_{\epsilon, r} \subset X$ such that $T(\epsilon)F(s, \psi, x) \in W^2_{\epsilon, r}$ for every $s \in [0, T]$, $\psi \in B_r[0, B]$ and $x \in B_r[0, X]$. Then there exists a mild solution of the initial value problem (1).

3. EXAMPLE

We consider briefly the Example 2.1 in [8]. Let $B$ be the space $C^r \times L^2(g; X)$, see Example 1.1 or [10], with $r = 0$ and $X = L^2([0, \pi])$. In this case $H = 1$; $M(t) = \gamma (-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^{t} g(\tau) d\tau\right)^{1/2}$ for all $t \geq 0$. Now, we consider the boundary value problem

$$\frac{\partial}{\partial t}[u(t, \xi) + \int_{-\infty}^{t} \int_{-\pi}^{\pi} b(s - t, \eta, \xi) u(s, \eta) d\eta ds] = \frac{\partial^2}{\partial \xi^2} u(t, \xi) + a_0(\xi) u(t, \xi) + \int_{-\infty}^{t} a(s - t) u(s, \xi) ds, \quad t \geq 0, \quad 0 \leq \xi \leq \pi,$$

(20)

$$u(t, 0) = u(t, \pi) = 0, \quad t \geq 0,$$

(21)

$$u(\tau, \xi) = \varphi(\tau, \xi), \quad \tau \leq 0, \quad 0 \leq \xi \leq \pi.$$

(22)

In addition to conditions (i)–(iv) in [8], we assume that

(v') The function $a(\cdot) \in L^\infty(\mathbb{R})$; $\frac{\partial}{\partial \eta} b(s, \eta, \xi)$ is measurable and

$$\sup_{t \in [0, T]} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\frac{\partial b}{\partial s}(s - t, \eta, \xi)\right)^2 d\eta ds d\xi < \infty,$$

$$\sup_{t \in [0, T]} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left(\frac{\partial b}{\partial s}(s - t, \eta, \xi)\right)^2 g(s)^{-1} d\eta ds d\xi < \infty.$$

Under these conditions, we define $G : [0, \infty) \times B \to X$ and $F : [0, \infty) \times B \times X \to X$ by

$$G(t, \psi)(\xi) := \int_{-\infty}^{0} \int_{-\pi}^{\pi} b(s, \eta, \xi) \psi(s, \eta) d\eta ds,$$

$$F(t, \psi, x)(\xi) := a_0(\xi) \psi(0, \xi) + \int_{-\infty}^{t} a(s) \psi(s, \xi) ds + x(\xi).$$

With the previous notation, we can represent (20)–(22) as the abstract Cauchy problem

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t, \int_{0}^{t} h(t, s, x_s) ds), \quad t \geq 0,$$

(23)
where $h : I \times I \times \mathcal{B} \to X$, $h(t, s, \psi)(\xi) = a(s - t)\psi(0, \xi)$ and $A$ is the operator $Ax = x''$ with domain

$$D(A) := \{ f(\cdot) \in L^2([0, \pi]) : f''(\cdot) \in L^2([0, \pi]), \ f(0) = f(\pi) = 0 \}.$$ 

It’s well known that $A$ is the infinitesimal generator of an analytic and compact semigroup on $X$. Moreover, see [8], $G([0, T] \times \mathcal{B}) \subseteq D((-A)^{1/2})$ and $\| (-A)^{1/2}G(t, \psi) \| \leq N_1 \| \psi \|_\mathcal{B}$, for every $(t, \psi) \in [0, T] \times \mathcal{B}$ where

$$N_1 := \int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{g(s)} \left( \frac{\partial}{\partial \zeta} b(s, \eta, \zeta) \right)^2 \, d\eta ds d\zeta.$$ 

Now we will prove that condition $G_\varphi$ holds for all $\varphi \in \mathcal{B}$. Let $x : (-\infty, T] \to X$ such that $x_0 = \varphi$ and $x \in C([0, T] : X)$. For $h > 0$ and $t \in [0, T]$ we have that

$$G(t + h, x_{t+h})(\xi) - G(t, x_t)(\xi) = \int_{-\infty}^t \int_0^\pi (b(s - t - h, \eta, \xi) - b(s - t, \eta, \xi)) \, x(s, \eta) \, d\eta ds$$

$$+ \int_t^{t+h} \int_0^\pi b(s - t - h, \eta, \xi) \, x(s, \eta) \, d\eta ds,$$

then

$$\left| \frac{d}{dt} G(t, x_t)(\xi) \right| \leq \int_{-\infty}^t \int_0^\pi \left| \frac{\partial b}{\partial s}(s - t, \eta, \xi) x(s, \eta) \right| \, d\eta ds + \int_0^\pi \left| b(0, \eta, \xi) x(t, \eta) \right| \, d\eta$$

$$\leq \| \varphi \|_\mathcal{B} \left( \int_{-\infty}^0 \int_0^\pi \left( \frac{\partial b}{\partial s}(s - t, \eta, \xi) \right)^2 g^{-1}(s) \, d\eta ds \right)^{1/2}$$

$$+ \int_0^t \left( \int_0^\pi \left( \frac{\partial b}{\partial s}(s - t, \eta, \xi) \right)^2 d\eta \right)^{1/2} \| x(s) \| \, ds$$

$$+ \left( \int_0^\pi b^2(0, \eta, \xi) d\eta \right)^{1/2} \| x(t) \|,$$

thus

$$\left\| \frac{d}{dt} G(t, x_t) \right\|_{L^2}^2 \leq 4 \| \varphi \|_\mathcal{B}^2 \int_{-\infty}^0 \int_0^\pi \int_0^\pi \left( \frac{\partial b}{\partial s}(s - t, \eta, \xi) \right)^2 g(s)^{-1} \, d\eta ds d\xi$$

$$+ 4 \| x \|_{T^2}^2 T \int_0^\pi \int_0^\pi \int_0^\pi \left( \frac{\partial b}{\partial s}(s - t, \eta, \xi) \right)^2 d\eta ds d\xi$$

$$+ 4 \| x \|_{T^2}^2 \int_0^\pi \int_0^\pi b^2(0, \eta, \xi) d\eta d\xi,$$

where $\| x \|_T := \sup \{ \| x(s) \| : s \in [0, T] \}$. This implies that the set of functions

$$\{ t \mapsto G(t, x_t) : x : (-\infty, T] \to X, x_0 = \varphi, x \in C([0, T] : X), \| x \|_T \leq r \}$$
is equicontinuous on $[0, T]$ for every $r > 0$. Thus, condition $G_{\varphi}$ holds for every $\varphi \in \mathcal{B}$.

On the other hand, for $\psi \in \mathcal{B}$ and $t \geq 0$ we have that

\[
\| F(t, \psi, x) \|_{L^2} \leq (\| a_0(\cdot) \|_{\infty} + \left( \int_{-\infty}^{-t} a^2(s) g(s)^{-1} ds \right)^{\frac{1}{2}}) \| \psi \|_{\mathcal{B}} + \| x \|_{L^2},
\]

\[
\| h(t, s, \psi) \|_{L^2} \leq \| a(\cdot) \|_{\infty} \| \psi \|_{\mathcal{B}} \Omega_h(\| \psi \|_{\mathcal{B}}).
\]

In relation with Theorem 2.1, it’s easy to confer that $c_1 = N_1$. Moreover, we can assume that $C > 0$ and that $m_F(s) = (\| a_0(\cdot) \|_{\infty} + \left( \int_{-\infty}^{-t} a^2(s) g(s)^{-1} ds \right)^{\frac{1}{2}} + 1); m_h(s) = \| a(\cdot) \|_{\infty} \Omega_h(s) = s$ and $\Omega_F(s) = s$.

Now, we establish without proof the following result.

**Corollary 3.1.** Let $T$ be a positive number and assume that condition $(b)$ in Theorem 2.1 holds. Then there exists a mild solution of the initial value problem (1).

**REFERENCES**


