From convex feasibility to convex constrained optimization using block action projection methods and underrelaxation

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Abstract

We describe the evolution of projection methods for solving convex feasibility problems to optimization methods when inconsistency arises, finally deriving from them, in a natural way, a general block method for convex constrained optimization. We present convergence results.

Keywords: projection methods; convex feasibility; constrained optimization

1. Introduction

We aim at solving the problem

$$\arg \min f(x) := \sum_{l=1}^{m} f_l(x),$$

s.t. : $x \in C = \cap_{i=1}^{n} C_i$.

where the function $f$ can be decomposed as a sum of $m$ convex differentiable functions, $f_l$, $l = 1, \ldots, m$ and $C_i$, $i = 1, \ldots, n$ are closed convex sets. The goal is to develop methods that take advantage of the function decomposition and of the separation of the constraints, using them one at a time as block action methods (blocks of the function and blocks of constraints). In that sense, our idea generalizes and extends a huge family of methods that started essentially with sequences of projections onto convex sets (POCS). The idea consists of sequentially projecting onto the convex sets and also sequentially performing gradient direction steps conveniently underrelaxed as shown in (7).

The method of orthogonal POCS has been widely used in many areas of application where the mathematical model is a convex feasibility problem (Bauschke and Borwein, 1996; Censor, 1981). When the problem is not feasible, the use of underrelaxation gives rise to an optimal solution in
the least squares sense (Censor et al., 1983, 2004), that is, an unconstrained convex optimization problem. A natural question that arises is about the behaviour of the method, under partial underrelaxation, when some of the convex sets do have an intersection, and they are the constraints of an optimization problem. The answer to this question is (7), a generalized block-action projection method slowly underrelaxed for solving constrained convex optimization problems.

In the next section we present the existing methods and results derived from POCS for convex feasibility, and its behavior when the feasible set is empty together with some remaining conjectures. As a natural consequence, in Section 3, we present the derivation of our projection-like method approach to convex optimization with convex constraints, where underrelaxation is used to push the generated sequence toward the optimum and just orthogonal projections are performed to ensure asymptotic feasibility. That is, the proposed algorithm combines orthogonal projections for the feasibility part and sequential underrelaxation for the search of the optimum. In Section 4 we present convergence results and a general proposal for future research appears in Section 5.

2. Convex feasibility and orthogonal projections

Our starting point is the solution of the following problem. Given \( n \) closed convex sets \( C_i \) \((i = 1, \ldots, n)\) (We will consider the sets in the finite dimensional Euclidean space \( \mathbb{R}^p \), but many of the results and conjectures could likely be generalized to a Hilbert space with some careful appropriate modifications and additional effort, as in (Bauschke and Edwards, 2005). It is worth noting that this generalization would not be easy):

\[
\text{Find} \quad x \in C = \cap_{i=1}^{n} C_i. \tag{2}
\]

The method that consists of sequentially computing the orthogonal projections (POCS) onto the convex sets is defined, for a given starting point \( x_0 \), by the sequence

\[
x_{k+1} = x_k + \lambda_k (P_{i_k} (x_k) - x_k), \tag{3}
\]

where \( P_{i}(x) \) denotes the orthogonal projection of the point \( x \) onto the set \( C_i \), \( \lambda_k \) is a sequence of positive relaxation parameters in the interval \((0,2)\) and \( i_k \) is a control sequence that we will consider cyclic in this article (but could be more general as in Lent and Censor, 1980). Under very general conditions (Gubin et al., 1962) and assuming that the intersection of the sets is non-empty, it can be proven that the sequence above converges to a feasible point (see Fig. 1).

![Sequential projections onto convex sets (POCS).](image)

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2.1. The feasibility problem without feasibility

But, what happens if the intersection set is empty? In that case, for fixed relaxation parameters, and, under tougher conditions (the most common ones are: one set is compact, the distances between sets are realized; De Pierro and Iusem, 1985; Gubin et al., 1962), it is possible to prove convergence of subsequences to $n$ different points (see Fig. 2). Many important results for this problem can be found in Bauschke and Borwein (1993) and Bauschke et al. (1997).

2.1.1. Approaching optimization

The feasibility problem without feasibility encounters finally an optimization problem. Let us consider the functions defined by $f_i(x) = \frac{1}{2}p_i\|x - x\|^2$, $i = 1, \ldots, n$. These are differentiable functions, their minima define the convex sets $C_i$ and their gradients are $-\nabla f_i(x) = p_i x - x$. So, in a general setting, we can consider the problem

$$\text{Minimize } f(x) := \sum_{i=1}^{n} f_i(x)$$

and POCS underrelaxed becomes

$$x_{k+1} = x_k - \lambda_k \nabla f_k(x_k).$$

So, now the algorithm becomes an alternative for general (differentiable) functions $f$ that can be decomposed as a sum of $f_i$’s. It was proven in Censor et al. (2004), that this algorithm converges under mild conditions for these general functions if the relaxation parameters satisfy

$$\lambda_k > 0, \quad \lambda_k \to 0, \quad \sum_{k=1}^{\infty} \lambda_k = \infty.$$  

Following De Pierro and Iusem (1985), it is probably easy to show that, for the underrelaxed POCS above, sufficient conditions for convergence of the whole sequence are, as for the parallel case, one compact set or realization of the distances.

![Fig. 2. POCS without feasibility.](image-url)
But, our main Conjecture (stated for the first time in De Pierro, 2001) is that “A necessary and sufficient condition for the convergence of Underrelaxed POCS is the existence of a least squares solution”.

3. Convex constrained optimization: an underrelaxed method

Analyzing carefully the behavior of POCS and its underrelaxed version, it is natural to derive a general algorithm for Convex Constrained Optimization, observing the fact that when there is feasibility, no (slow) underrelaxation is necessary. On the other hand, if a function has a minimum not contained in the constraints set, the direction of the negative gradient pushes the algorithm toward the minimum as a projection onto its level sets, but this direction should be conveniently (slowly!!) underrelaxed because we are looking for feasible points (see Fig. 3).

So, considering the previous motivation, the algorithm we propose for solving (1) is the following. Given a starting point $x_0$

$$
x_{k,0} := x_k$$

$$x_{k,l} := x_{k,l-1} - \lambda_k \nabla f_l(x_{k,l-1}), \quad l = 1, \ldots, m$$

$$x_{k+1} := P_{n} \cdots P_1(x_{k,m}),$$

where the sequence $(\lambda_k)$ satisfies (6). Essentially the algorithm consists of underrelaxed partial gradient steps followed by a sequence of orthogonal projections.

Fig. 3. Some sets intersect (constraints) and others do not (defining the decomposition of $f$).
The following is a well-known property (see Goebel and Reich, 1984, for a proof), that relates points and their orthogonal POCS, strongly needed for our proofs.

**Lemma 1.** For any \( y \in C \) and \( i = 1, \ldots, m \),

\[
\|P_i(x) - y\|^2 \leq \|x - y\|^2 - \|P_i(x) - x\|^2. \tag{8}
\]

Next we present a fundamental lemma that relates the behavior of the sequence and the decrease of the objective function.

**Lemma 2.** Let \( \{x_k\} \) be the sequence generated by (7) and suppose that the gradients are bounded, hence, there are scalars \( K_1, \ldots, K_m \) such that:

\[
\|\nabla f_i(x_{k,l-1})\| \leq K_l \quad l = 1, \ldots, m, \quad k = 0, 1, \ldots
\]

Then, for all \( y \in C \) we have

\[
\|x_{k+1} - y\|^2 \leq \|x_k - y\|^2 - 2\lambda_k [f(x_k) - f(y)] + \lambda_k^2 K^2,
\]

where \( K = \sum_{l=1}^m K_l \).

**Proof:** Just notice that Lemma 2.1 in Nedič and Bertsekas (2001) implies that, for all \( y \in C \), we have

\[
\|x_{k,m} - y\|^2 \leq \|x_k - y\|^2 - 2\lambda_k [f(x_k) - f(y)] + \lambda_k^2 K^2
\]

and apply (8) consecutively to get \( \|x_{k+1} - y\| \leq \|x_{k,m} - y\| \) for every \( y \in C \), and the claimed result holds. \( \blacksquare \)

**Remark:** As noted by a referee, boundedness of the gradients (or subgradients) is equivalent to the function be bounded on bounded sets (Proposition 7.8 of Bauschke and Borwein, 1996); also, it is worth noting (Example 7.11 in Bauschke and Borwein, 1996) that this equivalence does not remain valid in infinite dimensions.

### 4. Convergence analysis

Before starting, it will be useful to define:

\[
s_{k,0} := x_{k,m}; \quad s_{k,i} := P_i(s_{k,i-1}) = P_i P_{i-1} \cdots P_1(x_{k,m}), \quad i = 1, \ldots, n.
\]

For a given convex set \( C \), the orthogonal projection \( P \) onto it and a given point \( x \) defines the distance from \( x \) to \( C \),

\[
d_C(x) = \|P(x) - x\|.
\]

The next Proposition states that the sequence generated by (7) asymptotically approaches the feasible set. Assuming the sequence bounded, continuous differentiability implies that the sequences of the gradients are also bounded as required by Lemma 2. Before the proof of Proposition 1, we need the following auxiliary Lemma.
Lemma 3. For a given $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that, for every $y \in C$ and $x_{k,m}$ with $d_C(x_{k,m}) \geq \delta$, we have

$$\|x_{k+1} - y\|^2 \leq \|x_{k,m} - y\|^2 - \varepsilon_\delta,$$

implying,

$$d_C^2(x_{k+1}) \leq d_C^2(x_{k,m}) - \varepsilon_\delta.$$  \hspace{1cm} (12)

Proof: In order to prove the assertion we first notice that, by repeated iteration, (8) gives for every $y \in C$:

$$\|x_{k+1} - y\|^2 \leq \|x_{k,m} - y\|^2 - \sum_{i=1}^{n} \|P_i(s_{k,i-1}) - s_{k,i-1}\|^2$$

Now assume that (11) does not hold. In such a case, there exists a sequence $\{y_k\} \subset C$ and a subsequence $\{x_{k,m}\}$ such that $\|x_{k,m} - y_k\|^2 - \|x_{k+1} - y_k\|^2 \to 0$ and $d_C(x_{k,m}) \geq \delta$. Since $\{x_k\}$ and $\{\nabla f_i(x_{k,i-1})\}$ are bounded, $\{x_{k,m}\}$ is also bounded and we can assume $\{x_{h,m}\}$ to be convergent (taking subsequences if needed), say $\{x_{h,m}\} \to x^*$. Since $\|x_{h,m} - y_k\|^2 - \|x_{h+1} - y_k\|^2 \to 0$, the inequality above implies that $\|P_1(x_{h,m}) - x_{h,m}\|^2 \to 0$, which, in turn, gives $\|P_1(x^*) - x^*\|^2 = 0$, implying that $x^* \in C$.

Going further, remember that

$$\|s_{h,1} - s_{h,0}\| = \|P_1(s_{h,0}) - s_{h,0}\|.$$  

Taking into account that $\{s_{h,0}\} = \{x_{h,m}\} \to x^*$, this shows that $\{s_{h,1}\} \to x^*$. We can now repeat the same argument to show that $\|x_{h,m} - y_k\|^2 - \|x_{h+1} - y_k\|^2 \to 0$ implies $x^* \in C_2$ to obtain $\{s_{h,2}\} \to x^*$, and so on. This, of course, implies that $x^* \in C$, contradicting $d_C(x_{h,m}) \geq \delta$; so, (11) and (12) are true.  \[\blacksquare\]

Proposition 1. If the sequence $\{x_k\}$ generated by the algorithm (7) is bounded, then

$$\lim_{k \to \infty} d_C(x_k) = 0.$$

Proof: Now

$$\|x_{k,i} - x_{k,i-1}\| \leq \lambda_k \|\nabla f_i(x_{k,i-1})\|. \blacksquare$$

Summing up and in view of the boundedness of $\{\nabla f_i(x_{k,i-1})\}$, we see that there is $M > 0$ such that

$$\|x_k - x_{k,m}\| \leq \lambda_k M.$$  \hspace{1cm} (13)
On the other hand,
\[
\begin{align*}
d_C^2(x_{k,m}) &\leq \|x_{k,m} - x_k + x_k - P_C(x_k)\|^2 \\
&\leq (\|x_{k,m} - x_k\| + d_C(x_k))^2 \\
&\leq (\lambda_k M + d_C(x_k))^2 = d_C^2(x_k) + \lambda_k M(\lambda_k M + 2d_C(x_k)) \\
&\leq d_C^2(x_k) + \lambda_k N. \tag{14}
\end{align*}
\]

The first inequality follows from the definition of distance, the second is the triangle inequality, the third uses (12) and \( N \) in the fourth one is a bound of \( M(\lambda_k M + 2d_C(x_k)) \), that exists because the sequence is bounded.

Thus, if \( k \) is large enough such that \( \lambda_k M \leq \delta/2 \) and \( x_k \) is such that \( d_C(x_k) \geq \delta \) we clearly have \( d_C(x_{k,m}) \geq \delta/2 \), because of (12). So, for such large \( k \) we use the algorithm’s definition and (12), to get:
\[
d_C^2(x_{k+1}) \leq d_C^2(x_k) + \lambda_k N - \varepsilon\delta/2. \tag{15}
\]

Now let \( k \) be large enough such that \( \lambda_k N < \varepsilon\delta/2 \). For these large \( k \)'s, since \( d_C(x_k) \geq \delta \) we will have
\[
d_C^2(x_{k+1}) \leq d_C^2(x_k) - \varepsilon\delta/2. \tag{16}
\]

This clearly implies that we have infinitely many \( k \) such that \( d_C(x_k) < \delta \). Now we can argue in the following manner. First pick \( \bar{K} \) large enough such that, for \( k > \bar{K} \), the conditions needed to prove (15) and (16) hold (namely, that \( \lambda_k M \leq \delta/2 \) and \( \lambda_k N < \varepsilon\delta/2 \)) and also that \( d_C(x_{\bar{K}}) < \delta \). Now, for every \( k > \bar{K} \), we have the following possibilities:

1. \( d_C(x_k) < \delta \), which implies \( d_C(x_{k+1}) < \frac{3}{2} \delta \) (because, taking into account (12), in the worst case, \( x_{k,m} \) cannot be further from \( x_k \) than \( \delta/2 \), and \( x_{k+1} \) is closer to \( C \) than \( x_{k,m} \) by definition of the algorithm);

2. \( d_C(x_k) \geq \delta \), meaning \( d_C(x_{k+1}) < d_C(x_k) \) because of (16).

Going backwards up to \( \bar{K} \), the considerations above imply that, for \( k > \bar{K} \), \( d_C(x_k) < \frac{3}{2} \delta \) because \( d_C(x_{\bar{K}}) < \delta \). Once \( \delta > 0 \) was arbitrary, this proves our claim. \( \blacksquare \)

Consider now the optimal set \( X^* := \{x^* \in C | f(x^*) = f^* := \inf_{x \in C} f(x) \} \) and for a given \( \eta > 0 \) we define \( L_\eta := \{x \in C | f(x) \leq f^* + \eta \} \). We now present our main convergence result.

**Theorem 1.** If \( \{x_k\} \) is bounded and \( X^* \) is nonempty and bounded, then the sequence defined by the algorithm (7) satisfies
\[
d_{X^*}(x_k) \rightarrow 0, \quad f(x_k) \rightarrow f^*.
\]

**Proof:** Let \( \eta > 0 \), we then split the proof in two cases:

1. \( f(x_k) > f^* + \eta \). According to Lemma 2 we have, for every \( x^* \in X^* \)
\[
\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - 2\lambda_k [f(x_k) - f(x^*)] + \lambda_k^2 K^2.
\]
Since $\lambda_k \to 0$, we can assume $k$ large enough such that $\lambda_k K^2 \leq \eta$. For such large $k$ we will thus have

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \lambda_k \eta,$$

which gives (substituting $x^*$ by $P_{X^*}(x_k)$, that belongs to $X^*$)

$$d_{X^*}^2(x_{k+1}) \leq \|x_{k+1} - P_{X^*}(x_k)\|^2 \leq \|x_k - P_{X^*}(x_k)\|^2 - \lambda_k \eta$$

$$= d_{X^*}^2(x_k) - \lambda_k \eta. \quad (17)$$

This implies that

$$d_{X^*}(x_{k+1}) \leq d_{X^*}(x_k). \quad (18)$$

It is worth noting that the inequality (17) cannot be true $\forall k$ because the fact that $\sum_{k=1}^{\infty} \lambda_k = \infty$ would imply that the sequence is unbounded, as will be needed later. So, the next case will occur infinitely many times.

2. $f(x_k) \leq f^* + \eta$. Denote $\varepsilon_k := \|f(x_k) - f(P_C(x_k))\|$, then, continuity of $f$, boundedness of $\{x_k\}$ and the fact that $d_C(x_k) \to 0$ (From Proposition 1) imply that $\varepsilon_k \to 0$. Also, if we let $d(\eta) := \max_{x \in L_\eta} d_{X^*}(x)$, then boundedness of $L_\eta$ and continuity of $f$ can be used to verify that for $\eta, \varepsilon > 0$:

$$\lim_{\varepsilon \to 0} d(\eta + \varepsilon) = d(\eta). \quad (19)$$

($L_{\eta + \varepsilon}$ is a compact set, so, using the continuity and $u_\varepsilon = \arg \max_{x \in L_{\eta + \varepsilon}} d_{X^*}(x)$, it is easy to arrive to the desired result observing that any limit point should belong to $L_{\eta}$) Notice that $P_C(x_k) \in L_{\eta + \varepsilon}$, which gives $d_{X^*}(P_C(x_k)) \leq d(\eta + \varepsilon_k)$. On the other hand, using the triangular inequality and the non-expansiveness of projections we have:

$$d_{X^*}(x_k) = \|x_k - P_{X^*}(x_k)\| = \|x_k - P_{X^*}P_C(x_k) + P_{X^*}P_C(x_k) - P_{X^*}(x_k)\|$$

$$\leq \|x_k - P_{X^*}P_C(x_k)\| + \|P_{X^*}P_C(x_k) - P_{X^*}(x_k)\|$$

$$\leq \|x_k - P_{X^*}P_C(x_k)\| + \|P_C(x_k) - x_k\|$$

$$= \|x_k - P_C(x_k) + P_C(x_k) - P_{X^*}P_C(x_k)\| + d_C(x_k)$$

$$\leq \|x_k - P_C(x_k)\| + \|P_C(x_k) - P_{X^*}P_C(x_k)\| + d_C(x_k)$$

$$= d_{X^*}(P_C(x_k)) + 2d_C(x_k). \quad \blacksquare$$

This results in $d_{X^*}(x_k) \leq d(\eta + \varepsilon_k) + 2d_C(x_k)$. We can proceed to obtain the next sequence of inequalities (the first one uses the definition of distance, the second the triangle, the third one uses
Proof: We split in two cases:

1. \( x \)

If (9) holds, and \( P \) then we conclude (just by taking any convergent subsequence) that \( \limsup d_Y(x_k) \leq d(\eta) \).

If we think of (18) together with (20) (which must occur infinitely many times, because of (17) and \( \sum_{k=1}^{\infty} \lambda_k = \infty \)) we arrive at the conclusion that, for every \( k > K \) with \( K \) large enough, we have:

\[
\begin{align*}
\limsup_{k \to \infty} d_Y(x_k) & \leq \lim_{\eta \to 0} d(\eta)=0. \quad \blacksquare
\end{align*}
\]

We have used boundedness of \( \{x_k\} \) as a hypothesis to our demonstrations; now we give simple conditions under which the sequence generated by our algorithm is bounded.

**Proposition 2.** Suppose \( f \) is such that for some \( L > 0, \|x\| > L \Rightarrow f(x) > f(y) + \varepsilon \) for some \( y \in C \). If (9) holds, \( \lambda_k > 0 \) and \( \lambda_k \to 0 \) then \( \{x_k\} \), generated by (7), is bounded.

**Proof:** We split in two cases:

1. \( \|x\| > L \). Let \( \lambda_k \) be small enough such that \( \lambda_k K^2 < \varepsilon \), then we have, because of Lemma 2:

\[
\|x_{k+1} - y\|^2 \leq \|x_k - y\|^2 - \lambda_k \varepsilon.
\]

This implies that

\[
\|x_{k+1} - y\| \leq \|x_k - y\|.
\]

2. \( \|x_k\| \leq L \). In this case we have

\[
\|x_{k+1} - y\| \leq \|x_{k,m} - y\| \leq \sum_{i=1}^{m} \|x_{k,i} - x_{k,i-1}\| + \|x_k - y\| \leq \lambda_k K + M \| (using (9) for the first term of the inequality and \( M = L + \|y\| \)). \quad \blacksquare
\]

Now, if \( \bar{K} \) is such that for \( k > \bar{K} \), \( \lambda_k K^2 < \varepsilon \), application of the two cases above leads to

\[
\|x_k - y\| \leq \max\{\|x_0 - y\|, \ldots, \|x_{\bar{K}} - y\|, \sup\{M + \lambda_k K\} \} < \infty.
\]

This clearly implies that \( \{x_k\} \) is bounded. \quad \blacksquare

**5. Concluding remarks and future directions**

We have described a natural way of going from methods for solving large scale convex feasibility problems to convex constrained optimization that starts from sequences of orthogonal projections
and continues through underrelaxed gradient directions. Many problems remain open. Just as a sample: the parallel projections equivalent (Helou and De Pierro, unpublished data) to (7), the use of dual variables (Lent and Censor, 1980), optimization in two levels (once again no feasibility at all), the use of different metrics (De Pierro and Iusem, 1986; Browne and De Pierro, 1996; Helou and De Pierro, 2005) derived from Bregman distances or similar, approximate projections instead of exact ones, nondifferentiability (Helou and De Pierro, unpublished data), and their combinations.

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