Therefore $u_{tt} = c^2 u_{xx}$ and u is called a “weak” solution of the wave equation. In general, a *weak solution* of the wave equation is a distribution u for which

$$(u, \phi_{tt} - c^2 \phi_{xx}) = 0$$

for all test functions $\phi(x, t)$. □

Example 11.

Let S denote the sphere $\{|\mathbf{x}| = a\}$. Then the distribution $\phi \mapsto \iint_S \phi \, dS$ is denoted $\delta(|\mathbf{x}| - a)$. This notation makes sense because formally

$$\begin{aligned}\iiint \delta(|\mathbf{x}| - a)\phi(\mathbf{x})\, d\mathbf{x} &= \int_0^\infty \int_0^{2\pi} \int_0^\pi \phi(\mathbf{x}) \sin\theta \, d\theta \, d\psi \, \delta(r - a)\, r^2 dr \\ &= a^2 \int_0^{2\pi} \int_0^\pi \phi(\mathbf{x}) \sin\theta \, d\theta \, d\psi \\ &= \iint_S \phi \, dS.\end{aligned}$$

□

Example 12.

Let C be a smooth curve in space. Then the line integral over C defines the distribution $\phi \mapsto \int_C \phi \, ds$, where ds denotes the arc length. □

EXERCISES

1. Verify directly from the definition that $\phi \mapsto \int_{-\infty}^{\infty} f(x)\, \phi(x)\, dx$ is a distribution if $f(x)$ is any function that is integrable on each bounded set.
2. Let f be any distribution. Verify that the functional f' defined by $(f', \phi) = -(f, \phi')$ satisfies the linearity and continuity properties and therefore is another distribution.
3. Verify that the derivative is a linear operator on the vector space of distributions.
4. Denoting $p(x) = x^+$, show that $p' = H$ and $p'' = \delta$.
5. Verify, directly from the definition of a distribution, that the discontinuous function $u(x, t) = H(x - ct)$ is a weak solution of the wave equation.
6. Use Chapter 5 directly to prove (19) for all C^1 functions $\phi(x)$ that vanish near $\pm\pi$.
7. Let a sequence of L^2 functions $f_n(x)$ converge to a function $f(x)$ in the mean-square sense. Show that it also converges weakly in the sense of distributions.
8. (a) Show that the product $\delta(x)\delta(y)\delta(z)$ makes sense as a three-dimensional distribution.

(b) Show that $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$, where the first delta function is the three-dimensional one.

9. Show that no sense can be made of the square $[\delta(x)]^2$ as a distribution.
10. Verify that Example 11 is a distribution.
11. Verify that Example 12 is a distribution.
12. Let $\chi_a(x) = 1/2a$ for $-a < x < a$, and $\chi_a(x) = 0$ for $|x| > a$. Show that $\chi_a \to \delta$ weakly as $a \to 0$.

12.2 GREEN'S FUNCTIONS, REVISITED

Here we reinterpret the Green's functions and source functions for the most important PDEs.

LAPLACE OPERATOR

We saw in Section 6.1 that $1/r$ is a harmonic function in three dimensions except at the origin, where $r = |\mathbf{x}|$. Let $\phi(\mathbf{x})$ be a test function. By Exercise 7.2.2 we have the identity

$$\phi(\mathbf{0}) = -\iiint \frac{1}{r} \Delta\phi(\mathbf{x}) \frac{d\mathbf{x}}{4\pi}.$$

This means precisely that

$$\boxed{\Delta\left(-\frac{1}{4\pi r}\right) = \delta(\mathbf{x})} \tag{1}$$

in three dimensions. Because $\delta(\mathbf{x})$ vanishes except at the origin, formula (1) explains why $1/r$ is a harmonic function away from the origin and it explains exactly how it differs from being harmonic at the origin.

Consider now the Dirichlet problem for the Poisson equation,

$$\Delta u = f \quad \text{in } D, \qquad u = 0 \quad \text{on bdy } D.$$

Its solution is

$$u(\mathbf{x}_0) = \iiint_D G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x})\, d\mathbf{x} \tag{2}$$

from Theorem 7.3.2, where $G(\mathbf{x}, \mathbf{x}_0)$ is the Green's function. Now fix the point $\mathbf{x}_0 \in D$. The left side of (2) can be written as

$$u(\mathbf{x}_0) = \iiint_D \delta(\mathbf{x} - \mathbf{x}_0) u(\mathbf{x})\, d\mathbf{x}.$$

solution of the problem

$$\begin{aligned} S_t &= k\Delta S && \text{for } \mathbf{x} \in D \\ S &= 0 && \text{for } \mathbf{x} \in \text{bdy } D \\ S &= \delta(\mathbf{x} - \mathbf{x}_0) && \text{for } t = 0. \end{aligned} \tag{13}$$

We denote it by $S(\mathbf{x}, \mathbf{x}_0, t)$. Let $u(\mathbf{x}, t)$ denote the solution of the same problem but with the initial function $\phi(\mathbf{x})$. Let λ_n and $X_n(\mathbf{x})$ denote the eigenvalues and (normalized) eigenfunctions for the domain D, as in Chapter 11. Then

$$\begin{aligned} u(\mathbf{x}, t) &= \sum_{n=1}^{\infty} c_n e^{-\lambda_n kt}\, X_n(\mathbf{x}) \\ &= \sum_{n=1}^{\infty} \left[\iiint_D \phi(\mathbf{y}) X_n(\mathbf{y})\, d\mathbf{y} \right] e^{-\lambda_n kt} X_n(\mathbf{x}) \\ &= \iiint_D \left[\sum_{n=1}^{\infty} e^{-\lambda_n kt} X_n(\mathbf{x}) X_n(\mathbf{y}) \right] \phi(\mathbf{y})\, d\mathbf{y}, \end{aligned}$$

assuming that the switch of summation and integration is justified. Therefore, we have the formula

$$\boxed{S(\mathbf{x}, \mathbf{x}_0, t) = \sum_{n=1}^{\infty} e^{-\lambda_n kt} X_n(\mathbf{x}) X_n(\mathbf{x}_0).} \tag{14}$$

However, the convergence of this series is a delicate question that we do not pursue.

EXERCISES

1. Give an interpretation of $G(\mathbf{x}, \mathbf{x}_0)$ as a stationary wave or as the steady-state diffusion of a substance.
2. An infinite string, at rest for $t < 0$, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\,\delta(x - x_0)$, where V is a constant. Find the position of the string for $t > 0$.
3. A semi-infinite string ($0 < x < \infty$), at rest for $t < 0$ and held at $u = 0$ at the end, receives an instantaneous transverse blow at $t = 0$ which imparts an initial velocity of $V\,\delta(x - x_0)$, where V is a constant and $x_0 > 0$. Find the position of the string for $t > 0$.

4. Let $S(x, t)$ be the source function (Riemann function) for the one-dimensional wave equation. Calculate $\partial S/\partial t$ and find the PDE and initial conditions that it satisfies.
5. A force acting only at the origin leads to the wave equation $u_{tt} = c^2\Delta u + \delta(\mathbf{x})f(t)$ with vanishing initial conditions. Find the solution.
6. Find the formula for the general solution of the inhomogeneous wave equation in terms of the source function $S(\mathbf{x}, t)$.
7. Let $R(x, t) = S(x - x_0, t - t_0)$ for $t > t_0$ and let $R(x, t) \equiv 0$ for $t < t_0$. Let $R(x, t_0)$ remain undefined. Verify that R satisfies the inhomogeneous diffusion equation

$$R_t - k\,\Delta R = \delta(x - x_0)\delta(t - t_0).$$

8. (a) Prove that $\delta(a^2 - r^2) = \delta(a - r)/2a$ for $a > 0$ and $r > 0$.
(b) Deduce that the three-dimensional Riemann function for the wave equation for $t > 0$ is

$$S(\mathbf{x}, t) = \frac{1}{2\pi c}\delta(c^2t^2 - |\mathbf{x}|^2).$$

9. Derive the formula (12) for the Riemann function of the wave equation in two dimensions.
10. Consider an applied force $f(t)$ that acts only on the z axis and is independent of z, which leads to the wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}) + \delta(x, y)f(t)$$

with vanishing initial conditions. Find the solution.
11. For any $a \neq b$, derive the identity

$$\delta[(\lambda - a)(\lambda - b)] = \frac{1}{|a - b|}[\delta(\lambda - a) + \delta(\lambda - b)].$$

12. A rectangular plate $\{0 \le x \le a,\ 0 \le y \le b\}$ initially has a hot spot at its center so that its initial temperature distribution is $u(x, y, 0) = M\delta(x - \frac{a}{2}, y - \frac{b}{2})$. Its edges are maintained at zero temperature. Let k be the diffusion constant. Find the temperature at any later time in the form of a series.
13. Calculate the distribution $\Delta(\log r)$ in two dimensions.

12.3 FOURIER TRANSFORMS

Just as problems on finite intervals lead to Fourier series, problems on the whole line $(-\infty, \infty)$ lead to Fourier integrals. To understand this relationship, consider a function $f(x)$ defined on the interval $(-l, l)$. Its Fourier series, in

THREE DIMENSIONS

In three dimensions the Fourier transform is defined as

$$F(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x},$$

where $\mathbf{x} = (x, y, z)$, $\mathbf{k} = (k_1, k_2, k_3)$, and $\mathbf{k}\cdot\mathbf{x} = xk_1 + yk_2 + zk_3$. Then one recovers $f(\mathbf{x})$ from the formula

$$f(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k})\,e^{+i\mathbf{k}\cdot\mathbf{x}}\frac{d\mathbf{k}}{(2\pi)^3}.$$

EXERCISES

1. Verify each entry in the table of Fourier transforms. (Use (15) as needed.)
2. Verify each entry in the table of properties of Fourier transforms.
3. Show that

$$\frac{1}{2\pi^2 cr}\int_0^{\infty} \sin kct \sin kr\,dk = \frac{1}{8\pi^2 cr}\int_{-\infty}^{\infty}[e^{ik(ct-r)} - e^{ik(ct+r)}]\,dk$$
$$= \frac{1}{4\pi cr}[\delta(ct-r) - \delta(ct+r)].$$

4. Prove the following properties of the convolution.
   (a) $f * g = g * f$.
   (b) $(f * g)' = f' * g = f * g'$, where $'$ denotes the derivative in one variable.
   (c) $f * (g * h) = (f * g) * h$.
5. (a) Show that $\delta * f = f$ for any distribution f, where δ is the delta function.
   (b) Show that $\delta' * f = f'$ for any distribution f, where $'$ is the derivative.
6. Let $f(x)$ be a continuous function defined for $-\infty < x < \infty$ such that its Fourier transform $F(k)$ satisfies

$$F(k) = 0 \qquad \text{for } |k| > \pi.$$

   Such a function is said to be *band-limited*.
   (a) Show that

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin[\pi(x-n)]}{\pi(x-n)}.$$

   Thus $f(x)$ is completely determined by its values at the integers! We say that $f(x)$ is *sampled* at the integers.
   (b) Let $F(k) = 1$ in the interval $(-\pi, \pi)$ and $F(k) = 0$ outside this interval. Calculate both sides of (a) directly to verify that they are equal.
   (*Hints:* (a) Write $f(x)$ in terms of $F(k)$. Notice that $f(n)$ is the nth Fourier coefficient of $F(k)$ on $[-\pi, \pi]$. Deduce that $F(k) = \Sigma f(n)e^{-ink}$

in $[-\pi, \pi]$. Substitute this back into $f(x)$, and then interchange the integral with the series.)

7. (a) Let $f(x)$ be a continuous function on the line $(-\infty, \infty)$ that vanishes for large $|x|$. Show that the function

$$g(x) = \sum_{n=-\infty}^{\infty} f(x + 2\pi n)$$

is periodic with period 2π.

(b) Show that the Fourier coefficients c_m of $g(x)$ on the interval $(-\pi, \pi)$ are $F(m)/2\pi$, where $F(k)$ is the Fourier transform of $f(x)$.

(c) In the Fourier *series* of $g(x)$ on $(-\pi, \pi)$, let $x = 0$ to obtain the *Poisson summation formula*

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} F(n).$$

8. Let $\chi_a(x)$ be the function in Exercise 12.1.12. Compute its Fourier transform $\hat{\chi}_a(k)$. Use it to show that $\hat{\chi}_a \to 1$ weakly as $a \to 0$.
9. Use Fourier transforms to solve the ODE $-u_{xx} + a^2 u = \delta$, where $\delta = \delta(x)$ is the delta function.

12.4 SOURCE FUNCTIONS

In this section we show how useful the Fourier transform can be in finding the source function of a PDE *from scratch*.

DIFFUSION

The source function is properly defined as the unique solution of the problem

$$S_t = S_{xx} \quad (-\infty < x < \infty, \quad 0 < t < \infty), \qquad S(x, 0) = \delta(x) \qquad (1)$$

where we have taken the diffusion constant to be 1. Let's assume no knowledge at all about the form of $S(x, t)$. We only assume it has a Fourier transform as a distribution in x, for each t. Call its transform

$$\hat{S}(k, t) = \int_{-\infty}^{\infty} S(x, t) e^{-ikx}\, dx.$$

(Here k denotes the frequency variable, not the diffusion constant.) By property (i) of Fourier transforms, the PDE takes the form

$$\frac{\partial \hat{S}}{\partial t} = (ik)^2 \hat{S} = -k^2 \hat{S}, \qquad \hat{S}(k, 0) = 1. \qquad (2)$$

For each k this is an ODE that is easy to solve. The solution is

$$\hat{S}(k, t) = e^{-k^2 t}. \qquad (3)$$

This improper integral clearly converges for $y > 0$. It is split into two parts and integrated directly as

$$\begin{aligned} u(x, y) &= \frac{1}{2\pi(ix - y)} e^{ikx-ky}\Big|_0^\infty + \frac{1}{2\pi(ix + y)} e^{ikx+ky}\Big|_{-\infty}^0 \\ &= \frac{1}{2\pi}\left(\frac{1}{y - ix} + \frac{1}{y + ix}\right) = \frac{y}{\pi(x^2 + y^2)}, \end{aligned} \tag{17}$$

in agreement with Exercise 7.4.6.

EXERCISES

1. Use the Fourier transform directly to solve the heat equation with a convection term, namely, $u_t = \kappa u_{xx} + \mu u_x$ for $-\infty < x < \infty$, with an initial condition $u(x, 0) = \phi(x)$, assuming that $u(x, t)$ is bounded and $\kappa > 0$.
2. Use the Fourier transform in the x variable to find the harmonic function in the half-plane $\{y > 0\}$ that satisfies the Neumann condition $\partial u/\partial y = h(x)$ on $\{y = 0\}$.
3. Use the Fourier transform to find the bounded solution of the equation $-\Delta u + m^2 u = \delta(\mathbf{x})$ in free three-dimensional space with $m > 0$.
4. If $p(x)$ is a polynomial and $f(x)$ is any continuous function on the interval $[a, b]$, show that $g(x) = \int_a^b p(x - s) f(s)\, ds$ is also a polynomial.
5. In the three-dimensional half-space $\{(x, y, z) : z > 0\}$, solve the Laplace equation with $u(x, y, 0) = \delta(x, y)$, where δ denotes the delta function, as follows.
   (a) Show that
   $$u(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ikx+ily}\, e^{-z\sqrt{k^2+l^2}}\, \frac{dk\,dl}{4\pi^2}.$$
   (b) Letting $\rho = \sqrt{k^2 + l^2}$, $r = \sqrt{x^2 + y^2}$, and θ be the angle between (x, y) and (k, l), so that $xk + yl = \rho r \cos\theta$, show that
   $$u(x, y, z) = \int_0^{2\pi}\int_0^{\infty} e^{i\rho r\cos\theta} e^{-z\rho}\, \rho\, d\rho \frac{d\theta}{4\pi^2}.$$
   (c) Carry out the integral with respect to ρ and then use an extensive table of integrals to evaluate the θ integral.
6. Use the Fourier transform to solve $u_{xx} + u_{yy} = 0$ in the infinite strip $\{0 < y < 1, -\infty < x < \infty\}$, together with the conditions $u(x, 0) = 0$ and $u(x, 1) = f(x)$.