STABILITY

This is the third ingredient of well-posedness (see Section 1.5). It means that the initial and boundary conditions are correctly formulated. The energy method leads to the following form of stability of problem (3), in case $h = g = f = 0$. Let $u_1(x, 0) = \phi_1(x)$ and $u_2(x, 0) = \phi_2(x)$. Then $w = u_1 - u_2$ is the solution with the initial datum $\phi_1 - \phi_2$. So from (4) we have

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2\, dx \le \int_0^l [\phi_1(x) - \phi_2(x)]^2\, dx. \tag{5}$$

On the right side is a quantity that measures the nearness of the initial data for two solutions, and on the left we measure the nearness of the solutions at any later time. Thus, if we start nearby (at $t = 0$), we stay nearby. This is exactly the meaning of stability in the "square integral" sense (see Sections 1.5 and 5.4).

The maximum principle also proves the stability, but with a different way to measure nearness. Consider two solutions of (3) in a rectangle. We then have $w \equiv u_1 - u_2 = 0$ on the lateral sides of the rectangle and $w = \phi_1 - \phi_2$ on the bottom. The maximum principle asserts that throughout the rectangle

$$u_1(x,t) - u_2(x,t) \le \max|\phi_1 - \phi_2|.$$

The "minimum" principle says that

$$u_1(x,t) - u_2(x,t) \ge -\max|\phi_1 - \phi_2|.$$

Therefore,

$$\max_{0\le x\le l} |u_1(x,t) - u_2(x,t)| \le \max_{0\le x\le l} |\phi_1(x) - \phi_2(x)|, \tag{6}$$

valid for all $t > 0$. Equation (6) is in the same spirit as (5), but with a quite different method of measuring the nearness of functions. It is called stability in the "uniform" sense.

EXERCISES

1. Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \le x \le 1,\ 0 \le t \le T\}$.
2. Consider a solution of the diffusion equation $u_t = u_{xx}$ in $\{0 \le x \le l,\ 0 \le t < \infty\}$.
   (a) Let $M(T)$ = the maximum of $u(x, t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?
   (b) Let $m(T)$ = the minimum of $u(x, t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?
3. Consider the diffusion equation $u_t = u_{xx}$ in the interval (0, 1) with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.

(a) Show that $u(x, t) > 0$ at all interior points $0 < x < 1,\ 0 < t < \infty$.
(b) For each $t > 0$, let $\mu(t)$ = the maximum of $u(x, t)$ over $0 \le x \le 1$. Show that $\mu(t)$ is a decreasing (i.e., nonincreasing) function of t. (*Hint:* Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t), t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)
(c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.
(a) Show that $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.
(b) Show that $u(x, t) = u(1 - x, t)$ for all $t \ge 0$ and $0 \le x \le 1$.
(c) Use the energy method to show that $\int_0^1 u^2\, dx$ is a strictly decreasing function of t.

5. The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.
(a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2,\ 0 \le t \le 1\}$.
(b) Where precisely does our proof of the maximum principle break down for this equation?

6. Prove the *comparison principle* for the diffusion equation: If u and v are two solutions, and if $u \le v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \le v$ for $0 \le t < \infty,\ 0 \le x \le l$.

7. (a) More generally, if $u_t - ku_{xx} = f,\ v_t - kv_{xx} = g,\ f \le g$, and $u \le v$ at $x = 0$, $x = l$ and $t = 0$, prove that $u \le v$ for $0 \le x \le l$, $0 \le t < \infty$.
(b) If $v_t - v_{xx} \ge \sin x$ for $0 \le x \le \pi,\ 0 < t < \infty$, and if $v(0, t) \ge 0$, $v(\pi, t) \ge 0$ and $v(x, 0) \ge \sin x$, use part (a) to show that $v(x, t) \ge (1 - e^{-t}) \sin x$.

8. Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0, t) - a_0 u(0, t) = 0$ and $u_x(l, t) + a_l u(l, t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x, t)\, dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*.")

2.4 DIFFUSION ON THE WHOLE LINE

Our purpose in this section is to solve the problem

$$\boxed{\begin{aligned} u_t &= ku_{xx} \quad (-\infty < x < \infty,\ 0 < t < \infty) \\ u(x, 0) &= \phi(x). \end{aligned}} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

By the same reasoning as we used above, we end up with an explicit formula for $w(x, t)$. It is

$$\boxed{w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}]\, \phi(y)\, dy.} \tag{9}$$

This is carried out in Exercise 3. Notice that the only difference between (6) and (9) is a single minus sign!

Example 2.

Solve (7) with $\phi(x) = 1$. This is the same as Example 1 except for the single sign. So we can copy from that example:

$$u(x,t) = \left[\frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x}{4kt}\right)\right] + \left[\frac{1}{2} - \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x}{4kt}\right)\right] = 1.$$

(That was stupid: We could have guessed it!) □

EXERCISES

1. Solve $u_t = ku_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.
2. Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half-line $0 < x < \infty$.
3. Derive the solution formula for the half-line Neumann problem $w_t - kw_{xx} = 0$ for $0 < x < \infty$, $0 < t < \infty$; $w_x(0,t) = 0$; $w(x,0) = \phi(x)$.
4. Consider the following problem with a Robin boundary condition:

$$\begin{array}{lll} \text{DE:} & u_t = ku_{xx} & \text{on the half-line } 0 < x < \infty \\ & & \quad (\text{and } 0 < t < \infty) \\ \text{IC:} & u(x,0) = x & \text{for } t = 0 \text{ and } 0 < x < \infty \\ \text{BC:} & u_x(0,t) - 2u(0,t) = 0 & \text{for } x = 0. \end{array} \tag{$*$}$$

The purpose of this exercise is to verify the solution formula for (*). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?
(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?
(c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).
(d) Use Exercise 2.4.11 to show that w is an odd function of x.

The solution formula at any other point (x, t) is characterized by the number of reflections at each end ($x = 0, l$). This divides the space-time picture into diamond-shaped regions as illustrated in Figure 6. *Within each diamond the solution* $v(x, t)$ *is given by a different formula*. Further examples may be found in the exercises.

The formulas explain in detail how the solution looks. However, the method is impossible to generalize to two- or three-dimensional problems, nor does it work for the diffusion equation at all. Also, it is very complicated! Therefore, in Chapter 4 we shall introduce a completely different method (Fourier's) for solving problems on a finite interval.

EXERCISES

1. Solve the Neumann problem for the wave equation on the half-line $0 < x < \infty$.
2. The longitudinal vibrations of a semi-infinite flexible rod satisfy the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$. Assume that the end $x = 0$ is free ($u_x = 0$); it is initially at rest but has a constant initial velocity V for $a < x < 2a$ and has zero initial velocity elsewhere. Plot u versus x at the times $t = 0, a/c, 3a/2c, 2a/c$, and $3a/c$.
3. A wave $f(x + ct)$ travels along a semi-infinite string ($0 < x < \infty$) for $t < 0$. Find the vibrations $u(x, t)$ of the string for $t > 0$ if the end $x = 0$ is fixed.
4. Repeat Exercise 3 if the end is free.
5. Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0, t) = 0$, $u(x, 0) \equiv 1$, $u_t(x, 0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.
6. Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < \infty$, $0 \le t < \infty$, $u(x, 0) = 0$, $u_t(x, 0) = V$,
$$u_t(0, t) + a u_x(0, t) = 0,$$
where V, a, and c are positive constants and $a > c$.
7. (a) Show that $\phi_{\text{odd}}(x) = (\operatorname{sign} x)\phi(|x|)$.
   (b) Show that $\phi_{\text{ext}}(x) = \phi_{\text{odd}}(x - 2l[x/2l])$, where $[\cdot]$ denotes the greatest integer function.
   (c) Show that
$$\phi_{\text{ext}}(x) = \begin{cases} \phi\left(x - \left[\dfrac{x}{l}\right] l\right) & \text{if } \left[\dfrac{x}{l}\right] \text{ even} \\ -\phi\left(-x - \left[\dfrac{x}{l}\right] l - l\right) & \text{if } \left[\dfrac{x}{l}\right] \text{ odd.} \end{cases}$$
8. For the wave equation in a finite interval $(0, l)$ with Dirichlet conditions, explain the solution formula within each diamond-shaped region.

EXERCISES

1. Solve $u_{tt} = c^2 u_{xx} + xt$, $u(x, 0) = 0$, $u_t(x, 0) = 0$.
2. Solve $u_{tt} = c^2 u_{xx} + e^{ax}$, $u(x, 0) = 0$, $u_t(x, 0) = 0$.
3. Solve $u_{tt} = c^2 u_{xx} + \cos x$, $u(x, 0) = \sin x$, $u_t(x, 0) = 1 + x$.
4. Show that the solution of the inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x),$$

is the sum of three terms, one each for f, ϕ, and ψ.

5. Let $f(x, t)$ be any function and let $u(x, t) = (1/2c)\iint_\Delta f$, where Δ is the triangle of dependence. Verify directly by differentiation that

$$u_{tt} = c^2 u_{xx} + f \quad \text{and} \quad u(x, 0) \equiv u_t(x, 0) \equiv 0.$$

(*Hint:* Begin by writing the formula as the *iterated* integral

$$u(x, t) = \frac{1}{2c} \int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y, s)\, dy\, ds$$

and differentiate with care using the rule in the Appendix. This exercise is not easy.)

6. Derive the formula for the inhomogeneous wave equation in yet another way.
   (a) Write it as the system

$$u_t + cu_x = v, \quad v_t - cv_x = f.$$

   (b) Solve the first equation for u in terms of v as

$$u(x, t) = \int_0^t v(x - ct + cs, s)\, ds.$$

   (c) Similarly, solve the second equation for v in terms of f.
   (d) Substitute part (c) into part (b) and write as an iterated integral.

7. Let A be a positive-definite $n \times n$ matrix. Let

$$S(t) = \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m} t^{2m+1}}{(2m + 1)!}.$$

   (a) Show that this series of matrices converges uniformly for bounded t and its sum $S(t)$ solves the problem $S''(t) + A^2 S(t) = 0$, $S(0) = 0$, $S'(0) = I$, where I is the identity matrix. Therefore, it makes sense to denote $S(t)$ as $A^{-1} \sin tA$ and to denote its derivative $S'(t)$ as $\cos(tA)$.
   (b) Show that the solution of (13) is (14).

8. Show that the source operator for the wave equation solves the problem

$$\mathcal{S}_{tt} - c^2 \mathcal{S}_{xx} = 0, \quad \mathcal{S}(0) = 0, \quad \mathcal{S}_t(0) = I,$$

where I is the identity operator.

EXERCISES

1. (a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by one octave when the string is clamped exactly at its midpoint.
   (b) Explain why the note rises when the string is tightened.
2. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature $= 1$. Suddenly both ends are plunged into a bath of temperature $= 0$. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x, t)$ at later times. In this problem, *assume* the infinite series expansion
$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right)$$
3. A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, l)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0, l)$ with Dirichlet conditions at the ends. Separate the variables and use (8) to find its representation as a series.
4. Consider waves in a resistant medium that satisfy the problem
$$u_{tt} = c^2u_{xx} - ru_t \quad \text{for } 0 < x < l$$
$$u = 0 \quad \text{at both ends}$$
$$u(x, 0) = \phi(x) \quad u_t(x, 0) = \psi(x),$$
   where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.
5. Do the same for $2\pi c/l < r < 4\pi c/l$.
6. Separate the variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0, t) = u(\pi, t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x, 0) = 0$. So uniqueness is false for this equation!

4.2 THE NEUMANN CONDITION

The same method works for both the Neumann and Robin boundary conditions (BCs). In the former case, (4.1.2) is replaced by $u_x(0, t) = u_x(l, t) = 0$. Then the eigenfunctions are the solutions $X(x)$ of

$$\boxed{-X'' = \lambda X, \quad X'(0) = X'(l) = 0,} \tag{1}$$

other than the trivial solution $X(x) \equiv 0$.

As before, let's first search for the positive eigenvalues $\lambda = \beta^2 > 0$. As in (4.1.6), $X(x) = C\cos\beta x + D\sin\beta x$, so that

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x.$$

so that $T(t) = e^{-i\lambda t}$ and $X(x)$ satisfies exactly the same problem (1) as before. Therefore, the solution is

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-i(n\pi/l)^2 t} \cos\frac{n\pi x}{l}.}$$

The initial condition requires the cosine expansion (6).

EXERCISES

1. Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.
2. Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0,\ t) = 0$, $u(l, \text{t}) = 0$ (Neumann at the left, Dirichlet at the right).
   (a) Show that the eigenfunctions are $\cos[(n + \frac{1}{2})\pi x/l]$.
   (b) Write the series expansion for a solution $u(x,t)$.
3. Solve the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < l$ with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$.
4. Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \le x \le l$. Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx} \quad \text{for } -l \le x \le l$$
$$u(-l,t) = u(l,t) \quad \text{and} \quad u_x(-l,t) = u_x(l,t).$$

   These are called *periodic boundary conditions*.
   (a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$.
   (b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right) e^{-n^2\pi^2 kt/l^2}.$$

4.3 THE ROBIN CONDITION

We continue the method of separation of variables for the case of the Robin condition. The Robin condition means that we are solving $-X'' = \lambda X$ with the boundary conditions

$$\boxed{\begin{aligned} X' - a_0 X &= 0 \quad \text{at } x = 0 \\ X' + a_l X &= 0 \quad \text{at } x = l. \end{aligned}} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

The two constants a_0 and a_l should be considered as given.