2. Prove the uniqueness up to constants of the Neumann problem using the energy method.
3. Prove the uniqueness of the Robin problem $\partial u/\partial n + a(\mathbf{x})u(\mathbf{x}) = h(\mathbf{x})$ provided that $a(\mathbf{x}) > 0$ on the boundary.
4. Generalize the energy method to prove uniqueness for the diffusion equation with Dirichlet boundary conditions in three dimensions.
5. Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among *all* real-valued functions $w(\mathbf{x})$ on D the quantity

$$E[w] = \tfrac{1}{2}\iiint_D |\nabla w|^2\, d\mathbf{x} - \iint_{\text{bdy } D} hw\, dS$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$-\Delta u = 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial n} = h(\mathbf{x}) \quad \text{on bdy } D.$$

It is required to assume that the average of the given function $h(\mathbf{x})$ is zero (by Exercise 6.1.11).

Notice three features of this principle:

(i) There is *no constraint at all* on the trial functions $w(\mathbf{x})$.
(ii) The function $h(\mathbf{x})$ appears in the energy.
(iii) The functional $E[w]$ does not change if a constant is added to $w(\mathbf{x})$.

(*Hint:* Follow the method in Section 7.1.)

6. Let A and B be two disjoint bounded spatial domains, and let D be their exterior. So bdy D = bdy $A \cup$ bdy B. Consider a harmonic function $u(\mathbf{x})$ in D that tends to zero at infinity, which is *constant* on bdy A and *constant* on bdy B, and which satisfies

$$\iint_{\text{bdy } A} \frac{\partial u}{\partial n}\, dS = Q > 0 \qquad \text{and} \qquad \iint_{\text{bdy } B} \frac{\partial u}{\partial n}\, dS = 0.$$

[*Interpretation:* The harmonic function $u(\mathbf{x})$ is the electrostatic potential of two conductors, A and B; Q is the charge on A, while B is uncharged.]

(a) Show that the solution is unique. (*Hint:* Use the Hopf maximum principle.)
(b) Show that $u \geq 0$ in D. [*Hint:* If not, then $u(\mathbf{x})$ has a negative minimum. Use the Hopf principle again.]
(c) Show that $u > 0$ in D.

7. (*Rayleigh-Ritz approximation* to the harmonic function u in D with $u = h$ on bdy D.) Let $w_0, w_1, \ldots, w_n$ be arbitrary functions such that $w_0 = h$ on bdy D and $w_1 = \cdots = w_n = 0$ on bdy D. The problem is to find