Problems

All problems with blue boxes have an answer or hint available at the end of the book.

1. Give truth tables for the following expressions.

   a. $(s \vee t) \wedge (\neg s \vee t) \wedge (s \vee \neg t)$

   b. $(s \Rightarrow t) \wedge (t \Rightarrow u)$

   c. $(s \vee t \vee u) \wedge (s \vee \neg t \vee u)$

2. Find at least two more examples of the use of some word or phrase equivalent to "implies" in lemmas, theorems, or corollaries in Chapters 1 or 2.

3. Find at least two more examples of the use of the phrase "if and only if" in lemmas, theorems, and corollaries in Chapters 1 or 2.

4. Show that the statements $s \Rightarrow t$ and $\neg s \vee t$ are equivalent.

5. Prove the DeMorgan law that states $\neg(p \wedge q) = \neg p \vee \neg q$.

6. Show that $p \oplus q$ is equivalent to $(p \wedge \neg q) \vee (\neg p \wedge q)$.

7. Give a simplified form of each of the following expressions (using T to stand for a statement that is always true and F to stand for a statement that is always false).[4]

   a. $s \vee s$

   b. $s \wedge s$

   c. $s \vee \neg s$

   d. $s \wedge \neg s$

8. Using T to stand for a statement that is always true and F to stand for a statement that is always false, give a simplified form of each of the following statements.

   a. $\mathrm{T} \wedge s$

   b. $\mathrm{F} \wedge s$

   c. $\mathrm{T} \vee s$

   d. $\mathrm{F} \vee s$

[4] A statement that is always true is called a *tautology;* a statement that is always false is called a *contradiction.*

9. Use DeMorgan's law, the distributive law, and Problems 7 and/or 8 to show that

$$\neg(s \vee t) \vee \neg(s \vee \neg t)$$

is equivalent to $\neg s$.

10. Give an example in English where "or" seems to mean "exclusive or" (or where you think it would for many people) and an example in English where "or" seems to mean "inclusive or" (or where you think it would for many people).

11. Give an example in English where "if . . . then" seems to mean "if and only if" (or where you think it would to many people) and an example in English where it seems not to mean "if and only if" (or where you think it would not to many people).

12. Find a statement involving only $\wedge$, $\vee$, and $\neg$ (and s and t) equivalent to $s \Leftrightarrow t$. Does your statement have as few symbols as possible? If you think it doesn't, try to find one with fewer symbols.

13. Suppose that for each line of a two-variable truth table, you are told whether the final column in that line should evaluate to true or to false. (For example, you might be told that the final column should contain T, F, F, and T, in that order. Notice that Problem 12 can be interpreted as asking for this pattern.) Explain how to create a logical statement using the symbols s, t, $\wedge$, $\vee$, and $\neg$ that has that pattern as its final column. Can you extend this procedure to an arbitrary number of variables?

14. In Problem 13, your solution may have used $\wedge$, $\vee$, and $\neg$. Is it possible to give a solution using only one of these symbols? Is it possible to give a solution using only two of these symbols?

15. We proved that $\wedge$ distributes over $\vee$ in the sense of giving two equivalent statements that represent the two "sides" of the distributive law. Answer each question that follows, and explain why your answer is correct.

    a. Does $\vee$ distribute over $\wedge$?

    b. Does $\vee$ distribute over $\oplus$?

    c. Does $\wedge$ distribute over $\oplus$?

Problems

All problems with blue boxes have an answer or hint available at the end of the book.

1. For what positive integers x is the statement $(x-2)^2+1 \leq 2$ true? For what integers is it true? For what real numbers is it true? If you expand the universe for which you are considering a statement about a variable, does this always increase the size of the statement's truth set?

2. Is the statement "There is an integer greater than 2 such that $(x-2)^2+1 \leq 2$" true or false? How do you know?

3. Write the statement "The square of every real number is greater than or equal to 0" as a quantified statement about the universe of real numbers. You may use R to stand for the universe of real numbers.

4. A prime number is defined as an integer greater than 1 whose only positive integer factors are itself and 1. Find two ways to write this definition so that all quantifiers are explicit. (It may be convenient to introduce a variable to stand for the number and perhaps a variable or some variables for its factors.)

5. Write the definition of a greatest common divisor of m and n in such a way that all quantifiers are explicit and expressed explicitly as "for all" or "there exists." Write the part of Euclid's extended greatest common divisor theorem (Theorem 2.14) that relates the greatest common divisor of m and n algebraically to m and n. Again, make sure all quantifiers are explicit and expressed explicitly as "for all" or "there exists."

6. Using $s(x, y, z)$ to be the statement $x = yz$ and $t(x, y)$ to be the statement $x \leq y$, what is the form of the definition of a greatest common divisor d of m and n? (You need not include references to the universes for the variables.)

7. Which of the following statements (in which Z^+ stands for the positive integers and Z stands for all integers) is true and which is false? Explain why.

   a. $\forall z \in Z^+(z^2+6z+10 > 20)$

   b. $\forall z \in Z(z^2 - z \geq 0)$

   c. $\exists z \in Z^+(z - z^2 > 0)$

   d. $\exists z \in Z(z^2 - z = 6)$

8. Are there any (implicit) quantifiers in the statement "The product of odd integers is odd"? If so, what are they?

9. Rewrite the statement "The product of odd integers is odd" with all quantifiers (including any in the definition of odd integers) explicitly stated as "for all" or "there exist."

10. Rewrite the following statement without any negations: "There is no positive integer n such that for all integers $m > n$, all polynomial equations $p(x) = 0$ of degree m have no real numbers for solutions."

11. Consider the following slight modifications of Theorem 3.2. For each part, either prove that it is true or give a counterexample. Let U_1 be a universe, and let U_2 be another universe, with $U_1 \subseteq U_2$. Suppose that $q(x)$ is a statement about U_2 such that $U_1 = \{x \mid q(x) \text{ is true}\}$ and $p(x)$ is a statement about U_2.

    a. $\forall x \in U_1(p(x))$ is equivalent to $\forall x \in U_2(q(x) \wedge p(x))$.

    b. $\exists x \in U_1(p(x))$ is equivalent to $\exists x \in U_2(q(x) \Rightarrow p(x))$.

12. Let $p(x)$ stand for "x is a prime," $q(x)$ for "x is even," and $r(x, y)$ stand for "$x = y$." Use these three symbolic statements and appropriate logical notation to write the statement "There is one and only one even prime." (Use the set Z^+ of positive integers for your universe.)

13. Each of the following expressions represents a statement about the integers. Using $p(x)$ for "x is prime," $q(x, y)$ for "$x = y^2$," $r(x, y)$ for "$x \leq y$," $s(x, y, z)$ for "$z = xy$," and $t(x, y)$ for "$x = y$," determine which expressions represent true statements and which represent false statements.

    a. $\forall x \in Z(\exists y \in Z(q(x, y) \vee p(x)))$

    b. $\forall x \in Z(\forall y \in Z(s(x, x, y) \Leftrightarrow q(x, y)))$

    c. $\forall y \in Z(\exists x \in Z(q(y, x)))$

    d. $\exists z \in Z(\exists x \in Z(\exists y \in Z(p(x) \wedge p(y) \wedge \neg t(x, y))))$

14. Why is $(\exists x \in U(p(x))) \wedge (\exists y \in U(q(y)))$ not equivalent to $\exists z \in U(p(z) \wedge q(z))$? Are the statements $(\exists x \in U(p(x))) \vee (\exists y \in U(q(y)))$ and $\exists z \in U(p(z) \vee q(z))$ equivalent?

15. Give an example (in English) of a statement that has the form $\forall x \in U(\exists y \in V(p(x, y)))$. (The statement can be a mathematical statement, a statement about everyday life, or whatever you prefer.) Now write (in English) the statement using the same $p(x, y)$ but of the form $\exists y \in V(\forall x \in U(p(x, y)))$. Comment on whether "for all" and "there exist" commute.

3.3 INFERENCE

Direct Inference (Modus Ponens) and Proofs

In this section, we talk about the logical structure of proofs. The examples of proofs we give are chosen to illustrate a concept in a context that we hope will be familiar to you. These examples are not necessarily the only or the best way to prove the results. If you see other ways to do the proofs, that is good, because it means you are putting your prior knowledge to work. It would be useful to try to see how the ideas of this section apply to your alternate proofs.

Section 3.2 concluded with a proof that the sum of two even numbers is even. That proof contained several crucial ingredients. First, it introduced symbols for members of the universe of integers. In other words, rather than saying, "Suppose we have two integers," we used symbols for the two members of our universe by saying, "Let m and n be integers." How did we know to use algebraic symbols? There are many possible answers to this question. In this case, our intuition was probably based on thinking about what an even number is and realizing that the definition itself is essentially symbolic. (You may argue that an even number is just twice another number, and you would be right. Apparently there are no symbols [variables] in that definition. But they really are there in the phrases "even number" and "another number.") Because we all know algebra is easier with symbolic variables than with words, we should recognize that it makes sense to use algebraic notation. Thus, this decision was based on experience, not logic.

Next, we assumed the two integers were even. We then used the definition of even numbers; as our previous parenthetic comment suggests, it was natural to use the definition symbolically. The definition tells us that if m is an even number, then there exists an integer i such that $m = 2i$. We combined this with the assumption that m is even and concluded that, in fact, there does exist an integer i such that $m = 2i$. This argument is an example of using the principle of **direct inference** (called *modus ponens* in Latin).

5. *Contrapositive of* $p \Rightarrow q$. The contrapositive of the statement $p \Rightarrow q$ is the statement $\neg q \Rightarrow \neg p$.
6. *Converse of* $p \Rightarrow q$. The converse of the statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.
7. *Contrapositive rule of inference.* From $\neg q \Rightarrow \neg p$, we may conclude $p \Rightarrow q$.
8. *Principle of proof by contradiction.* If from assuming p and $\neg q$ we can derive both r and $\neg r$ for some statement r, then we may conclude $p \Rightarrow q$.

Problems

All problems with blue boxes have an answer or hint available at the end of the book.

1. Write the converse and contrapositive of each statement.
   a. If the hose is 60 ft long, then the hose will reach the tomatoes.
   b. George goes for a walk only if Mary goes for a walk.
   c. Pamela recites a poem if Andre asked for a poem.
2. Construct a proof that if m is odd, then m^2 is odd.
3. Construct a proof that for all integers m and n, if m is even and n is odd, then $m + n$ is odd.
4. What does it really mean to say, "Prove that if m is odd, and n is odd, then $m + n$ is even"? Prove this more precise statement.
5. Prove that for all integers m and n, if m is odd and n is odd, then mn is odd.
6. Is the statement $p \Rightarrow q$ equivalent to the statement $\neg p \Rightarrow \neg q$?
7. Construct a contrapositive proof that for all real numbers x, if $x^2 - 2x \neq -1$, then $x \neq 1$.
8. Construct a proof by contradiction that for all real numbers x, if $x^2 - 2x \neq -1$, then $x \neq 1$.
9. Prove that if $x^3 > 8$, then $x > 2$.
10. Prove that $\sqrt{3}$ is irrational.
11. Construct a proof that if m is an integer such that m^2 is even, then m is even.

12. Prove or disprove the following statement: "For every positive integer n, if n is prime, then 12 and $n^3 - n^2 + n$ have a common factor greater than 1."

13. Prove or disprove the following statement: "For all integers b, c, and d, if x is a rational number such that $x^2 + bx + c = d$, then x is an integer." (*Hints:* Are all the quantifiers given explicitly? It is okay, but not necessary, to use the quadratic formula.)

14. Prove that there is no largest prime number.

15. Prove that if f, g, and h are functions from R^+ to R^+ such that $f(x) = O(g(x))$ and $g(x) = O(h(x))$, then $f(x) = O(h(x))$.