Blow-up of solutions of semilinear heat equations with almost Hénon-critical exponent$^1$

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WENLU - João Pessoa, February 2018

$^1$Research partially supported by FAPESP/Brazil and Fondecyt/Chile.
The problem

We study the parabolic problem

\[
\begin{aligned}
    & u_t - \Delta u = |x|^\alpha |u|^{p-1}u & \text{in } B_1 \times (0, T) \\
    & u = 0 & \text{on } \partial B_1 \times (0, T) \\
    & u = u_0 & \text{in } B_1 \times \{0\},
\end{aligned}
\]

where

- $B_1$ is the unit ball in $\mathbb{R}^N$, $N \geq 3$;
- $T = T_{\text{max}}(u_0) \in (0, +\infty]$: the maximal existence time for the (classical) solution;
- $\alpha > 0$, $p > 1$;
- $u_0 \in C_0(B_1) := \{v \in C(\overline{B_1}): v = 0 \text{ on } \partial B_1\}$. 
Background: case $\alpha = 0$

Consider the problem

$$
\begin{cases}
    u_t - \Delta u = |u|^{p-1}u & \text{in } \Omega \times (0, T) \\
    u = 0 & \text{on } \partial \Omega \times (0, T) \\
    u = u_0 & \text{in } \Omega \times \{0\},
\end{cases}
$$

(P_0)

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $p > 1$, $u_0 \in \mathcal{C}_0(\Omega) = \{v \in C(\overline{\Omega}) : v(x) = 0 \text{ for } x \in \partial \Omega\}$.

- The well-posedness and several useful results (comparison, regularity..) for this equation can be found for example in (Quittner and Souplet, 2007).

- Our concern will be the study of the sets of initial data that produce global solutions or blow-up solutions.

$$
\mathcal{G} = \{u_0 \in \mathcal{C}_0(\Omega) : T_{\text{max}}(u_0) = \infty\} \\
\mathcal{F} = \{u_0 \in \mathcal{C}_0(\Omega) : T_{\text{max}}(u_0) < +\infty\}.
$$

$$
\mathcal{G}^+ = \{u_0 \in \mathcal{C}_0(\Omega) : u_0 \geq 0, T_{\text{max}}(u_0) = \infty\} \\
\mathcal{F}^+ = \{u_0 \in \mathcal{C}_0(\Omega) : u_0 \geq 0, T_{\text{max}}(u_0) < +\infty\}.
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Consider the problem

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\begin{align*}
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    u &= 0 & \text{on } \partial \Omega \times (0, T) \\
    u &= u_0 & \text{in } \Omega \times \{0\},
\end{align*}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N, N \geq 3, p > 1 \), \( u_0 \in C_0(\Omega) = \{ v \in C(\bar{\Omega}) : v(x) = 0 \text{ for } x \in \partial \Omega \} \).

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    \mathcal{G}^+ &= \{ u_0 \in C_0(\Omega) : u_0 \geq 0, T_{\text{max}}(u_0) = \infty \} \\
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\end{align*}
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  u_t - \Delta u = |u|^{p-1}u & \text{in } \Omega \times (0, T) \\
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where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $p > 1$, $u_0 \in C_0(\Omega) = \{v \in C(\overline{\Omega}) : v(x) = 0 \text{ for } x \in \partial\Omega\}$.

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\mathcal{G} = \{u_0 \in C_0(\Omega) : T_{\max}(u_0) = \infty\}
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$$
\mathcal{F}^+ = \{u_0 \in C_0(\Omega) : u_0 \geq 0, \ T_{\max}(u_0) < +\infty\}.
$$
Let \( w \in C_0(\Omega) \), \( w \not\equiv 0 \) and consider \( u_0 = \lambda w \), \( \lambda \geq 0 \):

- if \( \lambda \) is small enough then \( u_0 \in G \)
- if \( \lambda \) is large enough then \( u_0 \in F \)

Moreover, if \( w \geq 0 \), there exists \( \bar{\lambda} > 0 \) such that

- if \( 0 < \lambda < \bar{\lambda} \) then \( u_0 \in G \)
- if \( \lambda > \bar{\lambda} \) then \( u_0 \in F \)
- if \( \lambda = \bar{\lambda} \) both cases can occur

Thus, \( G^+ \) is star-shaped (in fact convex) with respect to 0.

When the initial value changes sign, the situation is different.

¿Is \( G \) star-shaped?
**Known properties of \((P_0)\)**

- Let \(w \in C_0(\Omega)\), \(w \not\equiv 0\) and consider \(u_0 = \lambda w\), \(\lambda \geq 0\):
  - if \(\lambda\) is small enough then \(u_0 \in \mathcal{G}\)
  - if \(\lambda\) is large enough then \(u_0 \in \mathcal{F}\)

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- Thus, \(\mathcal{G}^+\) is **star-shaped** (in fact convex) with respect to 0.
- When the initial value changes sign, the situation is different.
  ¿Is \(\mathcal{G}\) star-shaped?
First properties

Known properties of \((P_0)\)

- Let \(w \in C_0(\Omega), w \neq 0\) and consider \(u_0 = \lambda w, \lambda \geq 0\):
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- Thus, \(G^+\) is **star-shaped** (in fact convex) with respect to 0.
- When the initial value changes sign, the situation is different. ¿Is \(G\) star-shaped?
Let $\psi \in C_0(\Omega)$, $\psi \neq 0$ be a stationary solution and $u_0 = \lambda \psi$, $\lambda \geq 0$:

- if $\psi \geq 0$ (via comparison and energy arguments)
  - if $\lambda \leq 1$ then $u_0 \in G$
  - if $\lambda > 1$ then $u_0 \in F$

  OBS If $N = 1$ (or in symmetric situations) this is true even if $\psi$ changes sign

- if $\psi$ changes sign and $N > 1$ no easy argument:
  - if $\lambda$ is small enough then $u_0 \in G$
  - if $\lambda = 1$ then $u_0 \in G$
  - if $\lambda$ is large enough then $u_0 \in F$
First properties

**Known properties of (P₀)**

Let \( \psi \in C₀(\Omega) \neq 0 \) be a **stationary solution** and \( u₀ = \lambda \psi, \lambda \geq 0 \):

- if \( \psi \geq 0 \) (via comparison and energy arguments)
  - if \( \lambda \leq 1 \) then \( u₀ \in G \)
  - if \( \lambda > 1 \) then \( u₀ \in F \)

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- if \( \psi \) changes sign and \( N > 1 \) **no easy argument**:
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**Known properties of \((P_0)\)**

Let \(\psi \in C_0(\Omega) \psi \neq 0\) be a **stationary solution** and \(u_0 = \lambda \psi, \lambda \geq 0:\)

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Let \(\psi \in C_0(\Omega) \neq 0\) be a stationary solution and \(u_0 = \lambda \psi, \lambda \geq 0\):

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- if \(\psi\) changes sign and \(N > 1\) no easy argument:
  - if \(\lambda\) is small enough then \(u_0 \in G\)
  - if \(\lambda = 1\) then \(u_0 \in G\)
  - if \(\lambda\) is large enough then \(u_0 \in F\)
(Cazenave, Dickstein, and Weissler, 2009) consider radial solutions in $\Omega = B_1$:

**Theorem**

There exists $p^* < p_0 := \frac{N+2}{N-2}$ such that if $p^* < p < p_0$ and $\psi_p$ is a radial sign changing stationary solution of $(P_0)$, that is,

\[
\begin{align*}
-\Delta \psi_p &= |\psi_p|^{p-1}\psi_p & \text{in } B_1 \\
\psi_p &= 0 & \text{on } \partial B_1,
\end{align*}
\]

then there exists $\eta > 0$ such that

\[0 < |1 - \lambda| < \eta \Rightarrow u_0 = \lambda \psi_p \in \mathcal{F}\]

i.e. the solution of $(P_0)$, with $\Omega = B_1$ and $u_0 = \lambda \psi_p$, blows up in finite time both for $\lambda$ slightly greater than 1 and $\lambda$ slightly less than 1.

Hence $\mathcal{G}$ is not star-shaped with respect to the origin.
(Marino, Pacella, and Sciunzi, 2015) extended the previous result by considering a general bounded smooth domain $\Omega \subset \mathbb{R}^N$.

**Theorem**

There $\exists p^* < p_0 := \frac{N+2}{N-2}$ such that if $p^* < p < p_0$ and $\psi_p$ is a sign changing stationary solution of $(P_0)$ in $\Omega$, satisfying

\[
\int_{\Omega} |\nabla \psi_p|^2 \to 2S_0^{\frac{N}{2}} \quad \text{as} \quad p \to p_0 \tag{2.1}
\]

\[
\frac{\max \psi_p}{\min \psi_p} \to -\infty \quad \text{as} \quad p \to p_0. \tag{2.2}
\]

then there exists $\eta > 0$ such that

\[
0 < |1 - \lambda| < \eta \Rightarrow u_0 = \lambda \psi_p \in \mathcal{F}
\]

- Existence of solutions as above were proved in [(Pistoia and Weth, 2007; Musso and Pistoia, 2010)]
Sketch of the argument

The argument for both results is in three steps:

1. Let $\psi_p$ be a sign-changing stationary solution of $(P_0)$:
   - **step 3** (proved in (Gazzola and Weth, 2005)):
     - if $\exists t \geq 0$: $u(\cdot, t) \geq \psi_p$ then $u$ blows-up (positively).
     - if $\exists t \geq 0$: $u(\cdot, t) \leq \psi_p$ then it blows-up (negatively).
   - **step 2** (proved in (Cazenave, Dickstein, and Weissler, 2009)):
     - Proposition. Let $\varphi_{1,p}$ be a first eigenfunction of the linearized problem around $\psi_p$:

       $\left\{ \begin{array}{l}
       -\Delta \varphi - p|\psi_p|^{p-1}\varphi = \lambda \varphi \text{ in } \Omega \\
       \varphi = 0 \text{ on } \partial \Omega,
       \end{array} \right.$

       and assume that

       $\int_{\Omega} \psi_p \varphi_{1,p} > 0$.

       Then

       - for $\lambda > 1$ near 1, the solution $u_\lambda^\lambda$ of $(P_0)$ with initial value $u_0 = \lambda \psi_p$, satisfy $u_\lambda^\lambda(\cdot, t) \geq \psi_p$ for $t$ large enough.
       - for $\lambda < 1$ near 1, the solution $u_\lambda^\lambda$ of $(P_0)$ with initial value $u_0 = \lambda \psi_p$, satisfy $u_\lambda^\lambda(\cdot, t) \leq \psi_p$ for $t$ large enough.

   - **step 1**: prove that for $p < p_0$ near $p_0$

     $\int_{\Omega} \psi_p \varphi_{1,p} > 0$. 

References

CaDiWe - MaPaSc
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  \]

  and assume that

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Sketch of the argument

The argument for both results is in three steps:

1. let $\psi_p$ be a sign-changing stationary solution of $(P_0)$:
   - **step 3** (proved in (Gazzola and Weth, 2005)):
     - if $\exists t \geq 0$: $u(\cdot, t) \not\equiv \psi_p$ then $u$ blows-up (positively).
     - if $\exists t \geq 0$: $u(\cdot, t) \not\leq \psi_p$ then it blows-up (negatively).
   - **step 2**: (proved in (Cazenave, Dickstein, and Weissler, 2009)):
     - **Proposition.** let $\varphi_{1,p}$ be a first eigenfunction of the linearized problem around $\psi_p$:
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       \begin{cases}
       -\Delta \varphi - p|\psi_p|^{p-1} \varphi = \lambda \varphi & \text{in } \Omega \\
       \varphi = 0 & \text{on } \partial \Omega,
       \end{cases}
       \]
     - and assume that
       \[ \int_{\Omega} \psi_p \varphi_{1,p} > 0. \]
     - Then
       - for $\lambda > 1$ near 1, the solution $u_\lambda^p$ of $(P_0)$ with initial value $u_0 = \lambda \psi_p$, satisfy $u_\lambda^p(\cdot, t) \geq \psi_p$ for $t$ large enough.
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   - **step 1**: prove that for $p < p_0$ near $p_0$
     \[ \int_{\Omega} \psi_p \varphi_{1,p} > 0. \]
Blow-up mechanism

Stationary sign changing solution $\psi_p$ with $\int_\Omega \psi_p \varphi_{1,p} > 0$. 
Blow-up mechanism

Stationary sign changing solution $\psi_p$ with $\int_{\Omega} \psi_p \varphi_{1,p} > 0$. 

$$u_0 = 1^+ \psi_p$$

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$u_0 = 1^+ \psi_p$

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Stationary sign changing solution $\psi_p$ with $\int_\Omega \psi_p \varphi_{1,p} > 0$.

$u_0 = 1^+ \psi_p$

$u > \psi_p$

$u_0 = 1^- \psi_p$

$u < \psi_p$
Blow-up mechanism

Stationary sign changing solution $\psi_p$ with $\int_{\Omega} \psi_p \psi_1, \rho > 0$.

$u_0 = 1^+ \psi_p$

$u > \psi_p$

$u_0 = 1^- \psi_p$

$u < \psi_p$
Blow-up mechanism

Stationary sign changing solution $\psi_p$ with $\int_{\Omega} \psi_p \varphi_{1,p} > 0$.

$u_0 = 1^+ \psi_p$

$u > \psi_p$

$u_0 = 1^- \psi_p$

$u < \psi_p$
Stationary sign changing solution $\psi_p$ with $\int_\Omega \psi_p \varphi_{1,p} > 0$.

$u_0 = 1^+ \psi_p$ blew in finite time

$u_0 = 1^- \psi_p$ blew in finite time
The problem with $\alpha > 0$

We study the parabolic problem

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\begin{cases}
    u_t - \Delta u = |x|^\alpha |u|^{p-1}u & \text{in } B_1 \times (0, T) \\
    u = 0 & \text{on } \partial B_1 \times (0, T) \\
    u = u_0 & \text{in } B_1 \times \{0\},
\end{cases}
$$

(P$\alpha$)

where $B_1$ is the unit ball in $\mathbb{R}^N$, $N \geq 3$, $p > 1$, $\alpha > 0$.

- We restrict to $B_1$ and radial solutions because we need to work near the “relevant” critical exponent: $p_\alpha = \frac{N+2+2\alpha}{N-2}$

Actually,

$$H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \text{ for } 2 \leq p \leq p_0 + 1$$

but

$$H^1(\mathbb{R}^N) \not\hookrightarrow L^{p_\alpha+1}(\mathbb{R}^N), \quad H^1(B_1) \not\hookrightarrow L^{p_\alpha+1}(B_1, |x|^{\alpha})$$

$$H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, |x|^{\alpha}), \text{ for } 2 + \frac{\alpha}{N-1} \leq p \leq p_\alpha + 1$$
Problem \((P_\alpha)\) is **well posed** for \(u_0 \in C_0(B_1)\) (see (Wang, 1993)).

Classical results (comparison, energy, ..) still hold or can be adapted.

Step 3 was based on these methods then can be adapted.

Step 2 ...see below...

So we need to prove step 1:

**Proposition**

Given \(\alpha > 0\) there exists \(p^* > 0\) such that for each \(p \in (p^*, p_\alpha)\) the exists a radial sign-changing solution \(\psi_p \in C_0(B_1)\) of the elliptic problem

\[
\begin{align*}
-\Delta u &= |x|^\alpha |u|^{p-1}u \quad \text{in } B_1 \\
  u &= 0 \quad \text{on } \partial B_1,
\end{align*}
\]

such that, if \(\varphi_{1,p}\) is a first eigenfunction of the linearized problem around \(\psi_p\):

\[
\begin{align*}
-\Delta \varphi - p|x|^\alpha |\psi_p|^{p-1}|\varphi| &= \lambda \varphi \quad \text{in } B_1 \\
  \varphi &= 0 \quad \text{on } \partial B_1,
\end{align*}
\]

then

\[
\int_{B_1} \psi_p \varphi_{1,p} > 0.
\]
Problem (P_\(\alpha\)) is **well posed** for \(u_0 \in C_0(B_1)\) (see (Wang, 1993)).
- Classical results (*comparison, energy, ..*) still hold or can be adapted.
- **Step 3** was based on these methods then can be adapted.
- **Step 2** ...see below...
- So we need to prove **step 1**:

**Proposition**

*Given \(\alpha > 0\) there exists \(p^* > 0\) such that for each \(p \in (p^*, p_\alpha)\) the exists a radial sign-changing solution \(\psi_p \in C_0(B_1)\) of the elliptic problem*

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such that, if \(\varphi_{1,p}\) is a first eigenfunction of the linearized problem around \(\psi_p\):\[
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then

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\int_{B_1} \psi_p \varphi_{1,p} > 0.\]
Step 2

Let \( u_\lambda^\lambda \) be the solution of \((P_\alpha)\) with \( u_0 = \lambda \psi_p \) and define

\[
z_\lambda^\lambda(\cdot, t) = \frac{u_\lambda^\lambda(\cdot, t) - \psi_p}{\lambda - 1} \quad \text{in } B_1 \times (0, T).
\]

By continuous dependence, given \( 0 < \tau < T < \infty \), for \(|1 - \lambda| > 0\) small enough, \( u_\lambda^\lambda \) is well defined on \([0, T]\) and

\[
z_\lambda^\lambda \to z_p \quad \text{in } C([\tau, T], C^1(B_1)) \quad \text{as } \lambda \to 1,
\]

with \( z_p \) being a solution of the limiting problem

\[
\begin{cases}
(z_p)_t = \Delta z_p + p_\alpha |x|^\alpha |\psi_p|^{p_\alpha - 1} z_p & \text{in } B_1 \times (0, T) \\
z_p = 0 & \text{on } \partial B_1 \times (0, T) \\
z_p = \psi_p & \text{in } B_1 \times \{0\}.
\end{cases}
\]

But

\[
\int_{B_1} \psi_p \varphi_{1,p} > 0
\]

then at some \( t_0 > 0 \)

\[
z_\lambda^\lambda(\cdot, t_0) > 0 \quad \text{for } |\lambda - 1| \leq \delta.
\]
Step 2

Let $u_\lambda^p$ be the solution of $(P_\alpha)$ with $u_0 = \lambda \psi_p$ and define

$$z_\lambda^p(\cdot, t) = \frac{u_\lambda^p(\cdot, t) - \psi_p}{\lambda - 1} \quad \text{in } B_1 \times (0, T).$$

By continuous dependence, given $0 < \tau < T < \infty$, for $|1 - \lambda| > 0$ small enough, $u_\lambda^p$ is well defined on $[0, T]$ and

$$z_\lambda^p \to z_p \quad \text{in } C([\tau, T], C^1(B_1)) \quad \text{as } \lambda \to 1,$$

with $z_p$ being a solution of the limiting problem

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\begin{cases}
(z_p)_t = \Delta z_p + p_\alpha |x|^{\alpha} |\psi_p|^{p_\alpha - 1} z_p & \text{in } B_1 \times (0, T) \\
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\end{cases}
$$

But

$$\int_{B_1} \psi_p \varphi_{1,p} > 0$$

then at some $t_0 > 0$

$$z_\lambda^p(\cdot, t_0) > 0 \quad \text{for } |\lambda - 1| \leq \delta.$$
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- Let \( u_\lambda^\lambda \) be the solution of \((P_\alpha)\) with \( u_0 = \lambda \psi_p \) and define
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- By continuous dependence, given \( 0 < \tau < T < \infty \), for \(|1 - \lambda| > 0\) small enough, \( u_\lambda^\lambda \) is well defined on \([0, T]\) and
  \[
  z_\lambda^\lambda \rightarrow z_p \quad \text{in } C([\tau, T], C^1(B_1)) \quad \text{as } \lambda \rightarrow 1,
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  with \( z_p \) being a solution of the limiting problem
  \[
  \begin{aligned}
  (z_p)_t &= \Delta z_p + p_\alpha |x|^{\alpha} |\psi_p|^{p_\alpha - 1} z_p \quad \text{in } B_1 \times (0, T) \\
  z_p &= 0 \quad \text{on } \partial B_1 \times (0, T) \\
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  \end{aligned}
  \]
- But
  \[
  \int_{B_1} \psi_p \varphi_{1, p} > 0
  \]
  then at some \( t_0 > 0 \)
  \[
  z_\lambda^\lambda(\cdot, t_0) > 0 \quad \text{for } |\lambda - 1| \leq \delta.
  \]
Step 1 - The sign changing solutions

The radial sign-changing solutions \( \psi_p \) were found in (Alarcón, 2017) in the form

\[
\psi_p(x) = +PU_{M_1\varepsilon^{3/(N-2)},\alpha} - PU_{M_2\varepsilon^{1/(N-2)},\alpha} + \sigma_p(x) \quad x \in B_1,
\]

where

\[
U_{\lambda,\alpha}(x) = \gamma_{N,\alpha} \left( \frac{\lambda^{2+\alpha}}{\lambda^{2+\alpha} + |x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}}
\]

are the “bubbles of order \( \alpha \)”:

\[
\begin{cases}
-\Delta U = |x|^\alpha U^{\rho_{\alpha}} & \text{in } \mathbb{R}^N \\
U > 0 & \text{in } \mathbb{R}^N,
\end{cases}
\]

\( P \) is the projection on \( H^1(B_1) \);

\( \varepsilon = p_{\alpha} - p \), \( M_1, M_2 \) are positive constants depending only on \( N \) and \( \alpha \), and \( \sigma_p \) is a function which is of a lower order than the other terms (in \( C^1 \) norm) as \( p \uparrow p_{\alpha} \).
These solutions are called **Bubble towers**: the superposition of two bubbles that, as $p \nearrow p_\alpha$, concentrate at the origin at different speeds. They are obtained via the **Lyapunov-Schmidt finite dimensional reduction**.

They satisfy $\psi_p(0) > 0$,

$$\int_{B_1} |\nabla \psi_p|^2 \to 2S_\alpha^{\frac{N+\alpha}{2+\alpha}} \quad \text{as} \quad p \nearrow p_\alpha,$$

$$\max \psi_p, - \min \psi_p \to +\infty, \quad \frac{\max \psi_p}{\min \psi_p} \to -\infty \quad \text{as} \quad p \nearrow p_\alpha,$$
These solutions are called **Bubble towers**: the superposition of two bubbles that, as \( p \nearrow p_\alpha \), concentrate at the origin at different speeds.

They are obtained via the **Lyapunov-Schmidt finite dimensional reduction**.

They satisfy \( \psi_p(0) > 0 \),

\[
\int_{B_1} |\nabla \psi_p|^2 \to 2S_\alpha^{\frac{N+\alpha}{2+\alpha}} \quad \text{as} \quad p \nearrow p_\alpha, 
\]

\[
\max \psi_p, \quad -\min \psi_p \to +\infty, \quad \frac{\max \psi_p}{\min \psi_p} \to -\infty \quad \text{as} \quad p \nearrow p_\alpha, 
\]
These solutions are called **Bubble towers**: the superposition of two bubbles that, as $p \nearrow p_\alpha$, concentrate at the origin at different speeds.

- They are obtained via the **Lyapunov-Schmidt finite dimensional reduction**.
- They satisfy $\psi_p(0) > 0$, 
  $$\int_{B_1} |\nabla \psi_p|^2 \to 2S_\alpha^{\frac{N+\alpha}{2+\alpha}}$$  as $p \nearrow p_\alpha$, 
  
  $$\max \psi_p, - \min \psi_p \to +\infty, \quad \frac{\max \psi_p}{\min \psi_p} \to -\infty$$  as $p \nearrow p_\alpha$, 

![Graph showing the behavior of the solutions](image)
Final argument

Step 1

We compare these four problems:

\( \psi_p \) radial solution of
\[
\begin{cases}
-\Delta u = |x|^{\alpha}|u|^{p-1}u \\ u = 0
\end{cases}
\] in \( B_1 \)
\( u = 0 \) on \( \partial B_1 \),

\[ U \in \mathcal{D}^{1,2}_{rad}(\mathbb{R}^N) \] the radial positive solution of
\[
\begin{cases}
-\Delta U = |x|^{\alpha}U^{p\alpha} \\ U(0) = 1,
\end{cases}
\] in \( \mathbb{R}^N \)

\( \varphi_{1,p} \) is a radial first eigenfunction of
\[
\begin{cases}
-\Delta \varphi - p|x|^{\alpha}\psi_p|^{p-1}\psi = \lambda \varphi \\ \varphi = 0
\end{cases}
\] in \( B_1 \)
\( \varphi = 0 \) on \( \partial B_1 \),

\( \varphi^*_{1} \in H^1_{rad}(\mathbb{R}^N) \) is a first eigenfunction of
\[
-\Delta \varphi - p|x|^{\alpha}U^{p-1}\varphi = \lambda \varphi \] in \( \mathbb{R}^N \),
The rescaling

\[ M_p := \psi_p(0) = \|\psi_p\|_{L^\infty} \]

\[ \tilde{B}_p = M_p^{\frac{p-1}{2+\alpha}} B_1 \]

\[ \tilde{\psi}_p(x) := \frac{1}{M_p} \psi_p \left( \frac{x}{M_p^{\frac{p-1}{2+\alpha}}} \right) \quad \text{in } \tilde{B}_p \]

\[ \tilde{\varphi}_{1,p}(x) = \left( \frac{1}{M_p^{\frac{p-1}{2+\alpha}}} \right)^{\frac{N}{2}} \varphi_{1,p} \left( \frac{x}{M_p^{\frac{p-1}{2+\alpha}}} \right) \quad \text{in } \tilde{B}_p \]
The rescaling

Set

\[ M_p := \psi_p(0) = \|\psi_p\|_{L^\infty} \]

\[ \tilde{B}_p = M_p^{\frac{p-1}{2+\alpha}} B_1 \]

\[ \tilde{\psi}_p(x) := \frac{1}{M_p} \psi_p \left( \frac{x}{M_p^{\frac{p-1}{2+\alpha}}} \right) \quad \text{in} \quad \tilde{B}_p \]

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**The rescaling**

Set

\[ M_p := \psi_p(0) = \|\psi_p\|_{L^\infty} \]

\[ \tilde{B}_p = M_p^{\frac{p-1}{2+\alpha}} B_1 \]

\[ \tilde{\psi}_p(x) := \frac{1}{M_p} \psi_p \left( \frac{x}{M_p^{\frac{p-1}{2+\alpha}}} \right) \quad \text{in } \tilde{B}_p \]

\[ \tilde{\varphi}_{1,p}(x) = \left( \frac{1}{M_p^{\frac{p-1}{2+\alpha}}} \right)^{\frac{N}{2}} \varphi_{1,p} \left( \frac{x}{M_p^{\frac{p-1}{2+\alpha}}} \right) \quad \text{in } \tilde{B}_p \]
The rescaling

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\[ M_p := \psi_p(0) = \|\psi_p\|_{L^\infty} \]

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\[ \tilde{\psi}_p(x) := \frac{1}{M_p} \psi_p \left( \frac{x}{M_p^{\frac{p-1}{2+\alpha}}} \right) \quad \text{in} \quad \tilde{B}_p \]

\[ \tilde{\varphi}_{1,p}(x) = \left( \frac{1}{M_p^{\frac{p-1}{2+\alpha}}} \right)^{\frac{N}{2}} \varphi_{1,p} \left( \frac{x}{M_p^{\frac{p-1}{2+\alpha}}} \right) \quad \text{in} \quad \tilde{B}_p \]
Estimates on $\psi_p$ and $\tilde{\psi}_p$

One needs some estimates on the solutions $\psi_p$:

$$\max_{B_1} |x|^\alpha (\psi_p(x)^+)^{p-1} = O(\varepsilon^{-\frac{6}{N-2}})$$

$$\max_{B_1} |x|^\alpha (\psi_p(x)^-)^{p-1} = O(\varepsilon^{-\frac{2}{N-2}})$$

as $\varepsilon = p_\alpha - p \to 0$.

For the rescaled $\tilde{\psi}_p$ this implies

$$\max_{\tilde{B}_p} |x|^\alpha (\tilde{\psi}_p(x)^+)^{p-1} = O(1)$$

$$\max_{\tilde{B}_p} |x|^\alpha (\tilde{\psi}_p(x)^-)^{p-1} = O(\varepsilon^{4/(N-2)})$$

as $\varepsilon = p_\alpha - p \to 0$. 
Consider the Rayleigh functional

$$R^*(v) = \int_{\mathbb{R}^N} (|\nabla v|^2 - p_\alpha |x|^{\alpha} |U|^{p_\alpha - 1} v^2)$$

and define

$$\lambda_1^* := \inf_{v \in H^1_{\text{rad}}(\mathbb{R}^N), \|v\|_{L^2(\mathbb{R}^N)} = 1} R^*(v). \quad (3.4)$$

Then

- $-\infty < \lambda_1^* < 0$
- There exists a unique positive minimizer $\varphi_1^*$ associated to $\lambda_1^*$. 

Consider the Rayleigh functional

\[ R(v) = \int_{B_1} \left( |\nabla v|^2 - p|x|^\alpha |\psi_p|^{p-1}v^2 \right) \]

and define

\[ \lambda_{1,p} := \inf_{\substack{v \in H^1_{0,\text{rad}}(B_1) \\ \|v\|_{L^2(B_1)} = 1}} R(v). \quad (3.5) \]

Then

- \(-\infty < \lambda_{1,p} < 0\)
- There exists a unique positive minimizer \(\varphi_{1,p}\) associated to \(\lambda_{1,p}\).

After rescaling:

- \(\|\tilde{\varphi}_{1,p}\|_{L^2(\mathbb{R}^N)} = 1\)
- \(\tilde{\varphi}_{1,p}\) is the first eigenfunction of the following linearized problem:

\[
\begin{cases}
-\Delta \tilde{\varphi}_{1,p} - p|x|^\alpha |\tilde{\psi}_p|^{p-1} = \tilde{\lambda}_{1,p}\tilde{\varphi}_{1,p} & \text{in } \tilde{B}_p \\
\tilde{\varphi}_{1,p} = 0 & \text{on } \partial \tilde{B}_p ,
\end{cases}
\]

with \(\tilde{\lambda}_{1,p} = \frac{\lambda_{1,p}}{M_{p-1}}\).
Consider the Rayleigh functional

\[ R(v) = \int_{B_1} \left( |\nabla v|^2 - p|x|^\alpha |\psi_p|^{p-1} v^2 \right) \]

and define

\[ \lambda_{1,p} := \inf_{v \in H^1_{0,\text{rad}}(B_1)} \frac{R(v)}{\|v\|_{L^2(B_1)}} = 1 \]

Then

- \(-\infty < \lambda_{1,p} < 0\)
- There exists a unique positive minimizer \(\varphi_{1,p}\) associated to \(\lambda_{1,p}\).

After rescaling:

- \(\|\tilde{\varphi}_{1,p}\|_{L^2(\mathbb{R}^N)} = 1\)
- \(\tilde{\varphi}_{1,p}\) is the first eigenfunction of the following linearized problem:

\[
\begin{aligned}
-\Delta \tilde{\varphi}_{1,p} - p|x|^\alpha |\psi_p|^{p-1} &= \tilde{\lambda}_{1,p} \tilde{\varphi}_{1,p} & \text{in } \tilde{B}_p \\
\tilde{\varphi}_{1,p} &= 0 & \text{on } \partial \tilde{B}_p,
\end{aligned}
\]  

(3.6)

with \(\tilde{\lambda}_{1,p} = \frac{\lambda_{1,p}}{M_p^{2+\alpha}}\).
In order to conclude one has to prove:

- $\tilde{\psi}_p \to U$ in $C^2_{loc}(\mathbb{R}^N)$ ($U$ is the unique solution of the limiting problem).
- $\tilde{\lambda}_{1,p} \to \lambda_1^*$ (Several computations comparing the two minimization problems and using the properties of $\psi_p$)
- $\tilde{\varphi}_{1,p} \to \varphi_1^*$ in $L^2(\mathbb{R}^N)$ (follows from the previous, considering the minimizing sequence $\tilde{\varphi}_{1,p_n}$)

Finally,

- $\int_{B_1} \psi_p \varphi_{1,p}$, has the same sign as $\int_{B_1} |x|^{\alpha} |\psi_p|^{p-1} \psi_p \varphi_{1,p}$
- $\int_{B_1} |x|^{\alpha} |\psi_p|^{p-1} \psi_p \varphi_{1,p} = \int_{\tilde{B}_p} |x|^{\alpha} |\tilde{\psi}_p|^{p-1} \tilde{\psi}_p \tilde{\varphi}_{1,p} \to \int_{\mathbb{R}^N} |x|^{\alpha} U^{p\alpha} \varphi_1^* > 0$

**THEN**

\[ \int_{B_1} \psi_p \varphi_{1,p} > 0. \]
In order to conclude one has to prove:

- $\bar{\psi}_p \to U$ in $C^2_{loc}(\mathbb{R}^N)$ ($U$ is the unique solution of the limiting problem).
- $\bar{\lambda}_{1,p} \to \lambda_1^*$ (Several computations comparing the two minimization problems and using the properties of $\psi_p$)
- $\bar{\varphi}_{1,p} \to \varphi_1^*$ in $L^2(\mathbb{R}^N)$ (follows from the previous, considering the minimizing sequence $\bar{\varphi}_{1,p_n}$)

Finally,

- $\int_{B_1} \psi_p \varphi_{1,p}$, has the same sign as $\int_{B_1} |x|^\alpha |\psi_p|^{p-1} \psi_p \varphi_{1,p}$
- $\int_{B_1} |x|^\alpha |\psi_p|^{p-1} \psi_p \varphi_{1,p} = \int_{\tilde{B}_p} |x|^\alpha |\bar{\psi}_p|^{p-1} \bar{\psi}_p \bar{\varphi}_{1,p} \to \int_{\mathbb{R}^N} |x|^\alpha U^{p_\alpha} \varphi_1^* > 0$

THEN

$$\int_{B_1} \psi_p \varphi_{1,p} > 0.$$
In order to conclude one has to prove:

- \( \tilde{\psi}_p \to U \text{ in } C^2_{loc}(\mathbb{R}^N) \) (\( U \) is the unique solution of the limiting problem).
- \( \tilde{\lambda}_{1,p} \to \lambda^*_1 \) (Several computations comparing the two minimization problems and using the properties of \( \psi_p \))
- \( \tilde{\varphi}_{1,p} \to \varphi^*_1 \text{ in } L^2(\mathbb{R}^N) \) (follows from the previous, considering the minimizing sequence \( \tilde{\varphi}_{1,p_n} \))

Finally,

- \( \int_{B_1} \psi_p \varphi_{1,p}, \text{ has the same sign as } \int_{B_1} |x|^{\alpha} |\psi_p|^{p-1} \psi_p \varphi_{1,p} \)
- \( \int_{B_1} |x|^{\alpha} |\psi_p|^{p-1} \psi_p \varphi_{1,p} = \int_{\tilde{B}_p} |x|^{\alpha} |\tilde{\psi}_p|^{p-1} \tilde{\psi}_p \tilde{\varphi}_{1,p} \to \int_{\mathbb{R}^N} |x|^{\alpha} U^{p\alpha} \varphi^*_1 > 0 \)

THEN

\[ \int_{B_1} \psi_p \varphi_{1,p} > 0. \]
Theorem

There exists $p^* < p_\alpha = \frac{N+2+2\alpha}{N-2}$ with the following property:

If $p^* < p < p_\alpha$, then

\[ \exists \text{sign-changing radial stationary solution } \psi_p \text{ of } (P_\alpha) \text{ and } \delta_p > 0 \]

such that:

If $0 < |\lambda - 1| < \delta_p$, then the classical solution $u$ of $(P_\alpha)$ with initial value $u_0 = \lambda \psi_p$ blows up in finite time.

That is,

\[ 0 < |1 - \lambda| < \delta_p \Rightarrow u_0 = \lambda \psi_p \in \mathcal{F} \]

Then also for $(P_\alpha)$ the set $\mathcal{G}$ is not starshaped with respect to the origin.
Thank you very much for your attention.
Main references I

- **Alarcón, S. (2017).** “Multiple Sign changing solutions at the almost Hénon critical exponent”. In: *Preprint.*


- **Gazzola, F. and T. Weth (2005).** “Finite time blow-up and global solutions for semilinear parabolic equations with initial data at high energy level”. In: *Differential Integral Equations* 18.9, pp. 961–990.


Main references II

