1 The spectrum of the Laplacian

Given the **eigenvalue problem**

(Here Ω is a bounded domain in $\mathbb{R}^n$, and $Bu = 0$ represents Dirichlet or Neumann boundary conditions).

$$\begin{cases} -\Delta u = \lambda u & in\ \Omega \\ Bu = 0 & in\ \partial\Omega \end{cases} \tag{P_{eig}}$$

- The **spectrum of the Laplacian** is the set $\{\lambda_k\}_{k\in\mathbb{N}}$ of reals (**eigenvalues**) for which there exists a nontrivial solution.
- the corresponding nontrivial solutions (**eigenfunctions**) are denoted as $\phi_k,\ k \in \mathbb{N}$.

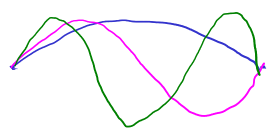
. . . .

Easy cases:

- (dimension 1 - Dirichlet)

$$\begin{cases} -u'' = \lambda u & in\ (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{P_{eig1}}$$

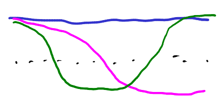
then $\lambda_k = k^2\pi^2,\ k = 1, 2, 3, ...,\ \phi_k(x) = \sin(k\pi\, x)$.



- (dimension 1 - Neumann)

$$\begin{cases} -u'' = \lambda u & in\ (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{P_{eig1}}$$

then $\lambda_k = k^2\pi^2,\ k = 0, 1, 2, 3, ...,\ \phi_k(x) = \cos(k\pi\, x)$.



- $\{\phi_k\}_{k\in\mathbb{N}}$ forms a very useful orthogonal basis in $L^2(\Omega)$ (and in $H := \big\{u:\ u, |\nabla u| \in L^2(\Omega)\big\}$).

  In particular it holds:

$$\int_\Omega |\nabla \phi_k|^2 = \lambda_k \int_\Omega |\phi_k|^2$$

$$\int_\Omega |\nabla u|^2 \leq \lambda_k \int_\Omega |u|^2\,, \qquad u \in span\,\{\phi_1, ..\phi_k\}$$

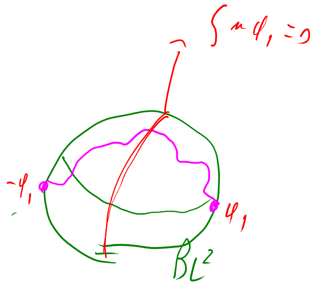
$$\int_\Omega |\nabla u|^2 \geq \lambda_{k+1} \int_\Omega |u|^2\,, \qquad u \in span\,\{\phi_1, ..\phi_k\}^\perp$$

Also,

$$\lambda_1 = \inf_{u \in H \cap S_{L^2}} \int_\Omega |\nabla u|^2$$

$$\lambda_2 = \inf_{u \in H \cap S_{L^2}, \int_\Omega u\phi_1 = 0} \int_\Omega |\nabla u|^2$$

$$= \inf_{\gamma \in \Gamma} \sup_{u \in Im(\gamma)} \int_\Omega |\nabla u|^2$$



where

$$\Gamma = \{\gamma \in \mathcal{C}([-1.1], S_{L^2}) : \ \gamma(\pm 1) = \pm \phi_1\} ,$$

in fact, there is a constrained critical point, satisfying $-\Delta u = \lambda u$ and $\lambda > \lambda_1$.

Let's show that the problem

$$\begin{cases} -\Delta u = \lambda u + \arctan(u) + 1 & in \ \ \Omega \\ u = 0 & in \ \ \partial\Omega \end{cases} . \tag{1.1}$$

has a solution if $\lambda_1 < \lambda < \lambda_2$:

Consider

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\lambda}{2} \int_\Omega u^2 - \int (Patn(u) + u) \, ;$$

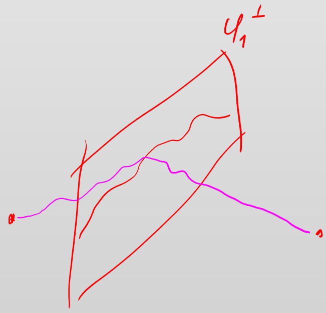
since $\lambda > \lambda_1$ it satisfies

$$\lim_{t \to \pm\infty} J(t\phi_1) = -\infty \, ;$$

since $\lambda < \lambda_2$

$$J|_{\phi_1^\perp} \geq C \, .$$

Then there exists a critical point at the level:



$$c = \inf_{\gamma \in \Gamma} \sup_{u \in Im(\gamma)} J(u)$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([-1.1], H) : \ \gamma(\pm 1) = \pm \phi_1\} \, .$$

Now let's try to show that the problem

$$\begin{cases} -u'' = \lambda u + e^u & in \ \ (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{Pexp}$$

has a solution if $\lambda_1 < \lambda < \lambda_2/4$.

Now

$$J(u) = \frac{1}{2}\int_0^1 |u''|^2 - \frac{\lambda}{2}\int_0^1 u^2 - \int_0^1 e^u - 1\,,$$

as before, it satisfies

$$\lim_{t\to\pm\infty} J(t\phi_1) = -\infty\,;$$

we need to prove that

$$\sup_{u\in Im(\gamma)} \ J \geq C\,;$$

for this we need to introduce the Fučík spectrum.

2 The Fučík spectrum

Given the problem

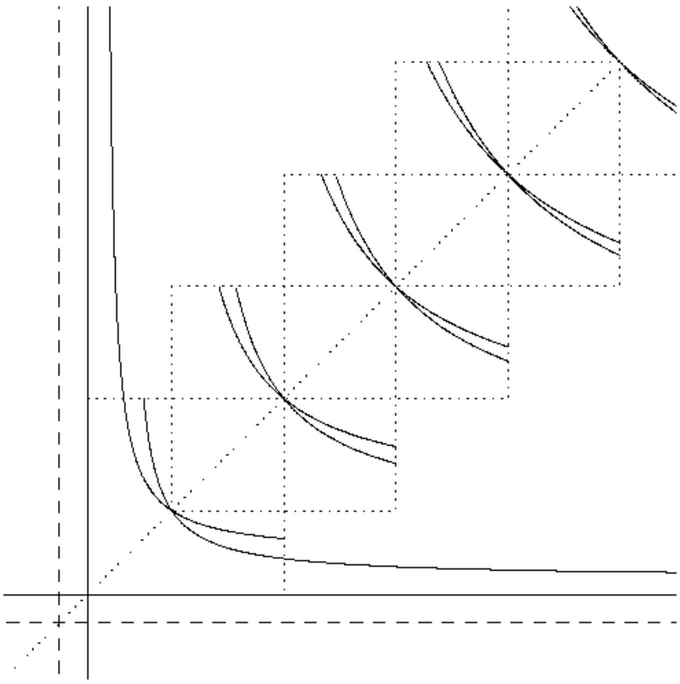
$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & in\ \ \Omega \\ Bu = 0 & in\ \ \partial\Omega \end{cases}, \tag{PF}$$

the **Fučík spectrum** (First introduced by Fučík and Dancer in 1976-77):

$$\Sigma = \left\{ (\lambda^+, \lambda^-) \in \mathbb{R}^2 \quad \text{such that (PF) has nontrivial solutions} \right\} .$$

(Here $u^\pm(x) = \max\{0, \pm u(x)\}$).

2.1 Fučík spectrum: PDE case



Known parts:

1. • trivial part: $\lambda^{\pm} = \lambda_1$ (since $\phi_1 > 0$), (λ_k, λ_k) (eigenvalues),
   • nontrivial part in $\lambda^{\pm} > \lambda_1$;
2. near the diagonal: two curves through (λ_k, λ_k) (in Gallouët-Kavian (81), Ruf (81), Magalhães (90))
3. first nontrivial curve (obtained variationally in de Figueiredo-Gossez (94)).

Given δ, let

$$\lambda = \inf_{\gamma \in \Gamma} \sup_{u \in Im(\gamma)} \int_\Omega |\nabla u|^2$$

where

$$\Gamma = \left\{ \gamma \in \mathcal{C}([-1.1], Q) : \ \gamma(-1) = -\phi_1/\sqrt{\delta}, \ \gamma(1) = \phi_1 \right\}$$

$$Q = \left\{ u \in H : \ \int_\Omega |u^+|^2 + \delta \int_\Omega |u^-|^2 = 1 \right\}.$$

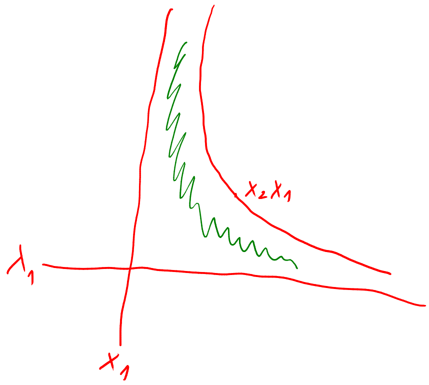
Then $(\lambda, \lambda\delta) \in \Sigma$.

In fact, there is a constrained critical point, satisfying $-\Delta u = \lambda(u^+ - \delta u^-)$ and $\lambda > \lambda_1$.

As before, with this we may prove existence for

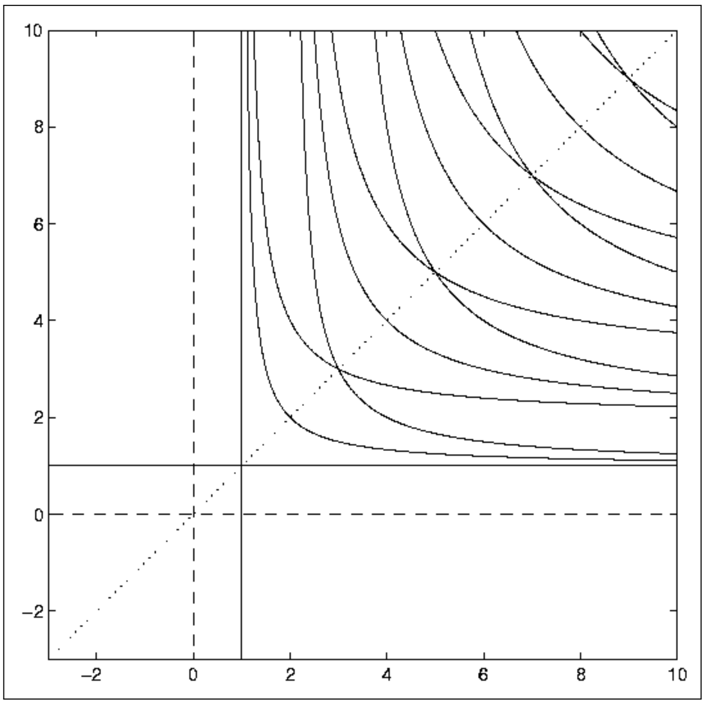
$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- + \arctan(u) + 1 & in \ \ \Omega \\ u = 0 & in \ \ \partial\Omega \end{cases} \tag{2.1}$$

if (λ^+, λ^-) is below this first nontrivial curve.

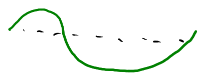


2.2 Fučík spectrum: ODE Dirichlet case

$$\begin{cases} -u'' = \lambda^+ u^+ - \lambda^- u^- \quad in \;\; (0,1) \\ u(0) = u(1) = 0 \end{cases} . \tag{2.2}$$

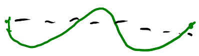


$$\Sigma_{2i} \; : \; \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda^+}} + \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda^-}} = 1$$

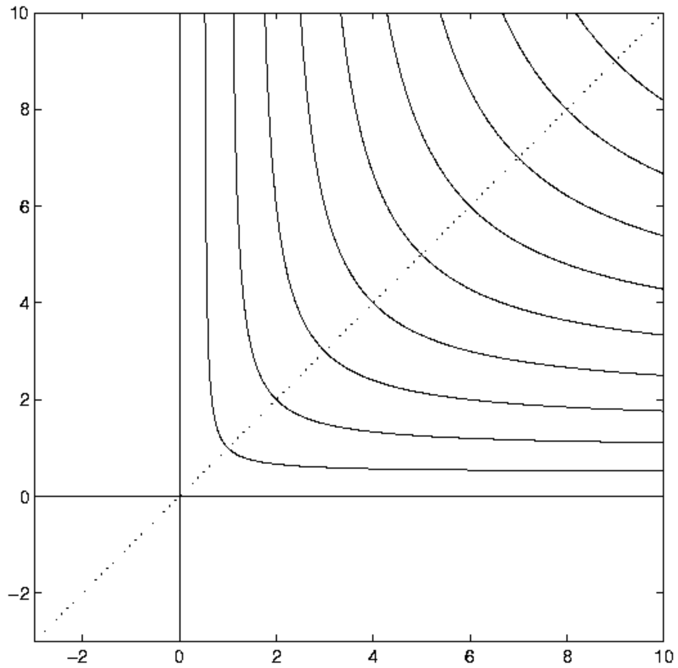


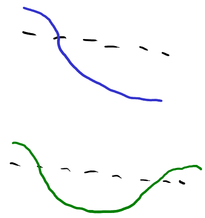
$$\Sigma^+_{2i-1} \; : \; \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda^+}} + \frac{(i-1)\sqrt{\lambda_1}}{\sqrt{\lambda^-}} = 1$$

$$\Sigma^-_{2i-1} \; : \; \frac{(i-1)\sqrt{\lambda_1}}{\sqrt{\lambda^+}} + \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda^-}} = 1$$



2.3 Fučík spectrum: ODE Neumann/Periodic case





$$\Sigma_k \ : \ \frac{(k-1)\sqrt{\lambda_2}}{2\sqrt{\lambda^+}} + \frac{(k-1)\sqrt{\lambda_2}}{2\sqrt{\lambda^-}} = 1$$

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The shape of Σ means we can prove existence for

$$\begin{cases} -u'' = \lambda^+ u^+ - \lambda^- u^- + \arctan(u) + 1 \quad in \ \ (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{2.3}$$

if (λ^+, λ^-) is below the first curve Σ_2.

in fact we can prove existence if $\lambda_1 < \lambda^- < \lambda_2/4$ and ANY $\lambda^+ > \lambda_1$.

Finally, we can use this to prove that

$$\begin{cases} -u'' = \lambda u + e^u \qquad in \ \ (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \quad . \tag{Pexp}$$

has a solution if $\lambda_1 < \lambda < \lambda_2/4$.

3 More in general

$$\begin{cases} -u'' = \lambda u + g(x,u) + h(x) \quad in \ (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{3.1}$$

- $g \in \mathcal{C}^0([0,1] \times \mathbb{R})$, $h \in L^2(0,1)$,
- $\lim_{s \to -\infty} \frac{g(x,s)}{s} = 0, \quad \lim_{s \to +\infty} \frac{g(x,s)}{s} = +\infty$ uniformly with respect to $x \in [0,1]$
- Some more Technical hypotheses to achieve PS condition

. . . .

Results:

- For $\lambda < \lambda_1$: **Ambrosetti-Prodi (72)**: 0-1-2 solutions depending on h.
- For $\lambda \in \left(\lambda_1, \frac{\lambda_2}{4}\right)$: de Figueiredo-Ruf (91), Villegas (98): existence $\forall\ h$
- ([with Periodic conditions, $\lambda \in \left(\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4}\right)$ existence $\forall h$: de Figueiredo-Ruf (93)])
- For $\lambda \in \left(\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4}\right)$ existence $\forall h$: M. (04)

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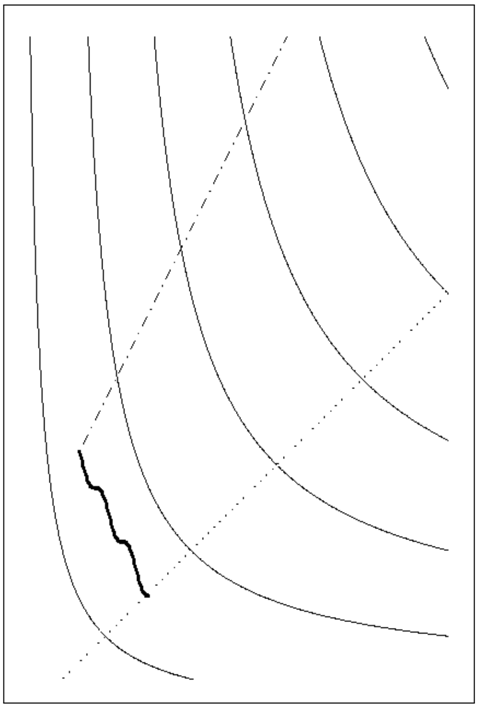
The result comes from three ingredients

- the knowledge of Σ
- the variational characterization of Σ
- the existence of a critical point, which is proved using the two items above!

3.1 A variational characterization of Σ



M. (04)

Teorema 3.1. *Let* $(\alpha^+, \alpha^-) \notin \Sigma$ *with* $\alpha^+ \geq \alpha^-$ *be such that* $\exists a \in (\lambda_k, \lambda_{k+1})$ *and a* $\mathcal{C}^1$ *function* $\alpha : [0,1] \to \mathbb{R}^2$ *such that:*

a) $\alpha(0) = (a, a)$, $\alpha(1) = (\alpha^+, \alpha^-)$;

b) $\alpha([0,1]) \cap \Sigma = \emptyset$.

Then we can find and characterize one intersection of the Fučík spectrum with the halfline $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$, *for each value of* $r \in (0,1]$.

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3.2 More on the Fucik Spectrum

- (Massa, 2004a): for the p-Laplacian
- (Massa and Ruf, 2006; Massa and Ruf, 2007): for systems
- (Massa and Ruf, 2009): a special case on a torus
- (Molle and Passaseo, 2014; Molle and Passaseo, 2015a; Molle and Passaseo, 2015b): more recent results

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