FUNCTIONAL DIFFERENTIAL EQUATIONS

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A NEW APPROACH TO IMPULSIVE RETARDED DIFFERENTIAL EQUATIONS: STABILITY RESULTS *

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Dedicated to the memory of Michael Drakhlin.

Abstract. It is known that impulsive retarded functional differential equations can be regarded as Banach-space valued generalized ordinary differential equations (generalized ODEs). In the present paper, we discuss the variational stability and variational asymptotic stability of the zero solution of generalized ODEs and we apply these results to obtain the Lyapunov uniform stability and uniform asymptotic stability for a large class of retarded differential equations with pre-assigned moments of impulse action, via Lyapunov functionals.

Key Words. functional differential equation, impulses, stability

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1. Introduction. Let $I \subset \mathbb{R}$ be an interval. If X is a Banach space, by $PC(I, X)$ we mean the space of functions $y : I \to X$ which are continuous from the left, that is, $y(t-) = y(t)$ and admit the right limits $y(t+)$. Note that $PC(I, X)$ is in fact the set of X-valued left continuous regulated functions on I.

By $|\cdot|$ we denote a norm in $\mathbb{R}^n$. If $I \subset \mathbb{R}$ is a compact interval, then for $\varphi \in PC(I, \mathbb{R}^n)$ we set $\|\varphi\| = \sup_{t \in I} |\varphi(t)|$. It is known that in this case $PC(I, \mathbb{R}^n)$ with the norm $\|\cdot\|$ forms a Banach space.

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Given $r \geq 0$ and a function $y : \mathbb{R} \to \mathbb{R}^n$, let $y_t : [-r, 0] \to \mathbb{R}^n$ be given by $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$, $t \in \mathbb{R}$. It is clear that for $a \in \mathbb{R}$ and a function $y \in PC([a-r, \infty), \mathbb{R}^n)$, we have $y_t \in PC([-r, 0], \mathbb{R}^n)$ for all $t \in [a, \infty)$.

For $y : \mathbb{R} \to \mathbb{R}^n$ belonging to $PC(\mathbb{R}, \mathbb{R}^n)$ put

$$\Delta y(t) := y(t+) - y(t-) = y(t+) - y(t) \text{ for } t \in \mathbb{R}.$$

If $b : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone increasing satisfying $b(0) = 0$, then we say that b is a function of *Hahn class*.

Let $r \geq 0$ and $0 \leq t_1 < \ldots < t_k < \ldots$, with $t_k \to +\infty$ as $k \to +\infty$.

An impulsive retarded functional differential equation is formally written in the form

$$\begin{cases} \dot{y}(t) = f(y_t, t), \; t \neq t_k \\ \Delta y(t_k) = I_k(y(t_k)), \; k = 1, 2, \ldots, \end{cases} \tag{1.1}$$

where $y \in \mathbb{R}^n \mapsto I_k(y) \in \mathbb{R}^n$. We will study equations of this type for the case $t \geq 0$.

Given an $a \geq 0$ and $\phi \in PC([-r, 0], \mathbb{R}^n)$, the initial value problem with impulses corresponding to the equation (1.1) and the given data has the form

$$\begin{cases} \dot{y}(t) = f(y_t, t), \; t \neq t_k \\ \Delta y(t_k) = I_k(y(t_k)), \; k = 1, 2, \ldots \\ y_a = \phi. \end{cases} \tag{1.2}$$

Without specifying the properties of the function f in the first relation in (1.1) or (1.2) at this moment, we suppose that a solution y belongs to PC on its interval of definition.

The equation (1.1) can be interpreted in a certain "integral" form

$$y(t) = y(0) + \int_0^t f(y_s, s)\, ds + \sum_{0 \leq t_k < t} I_k(y(t_k)) \text{ for } t > 0,$$

which comes from integrating the part without impulses and the "jumping" of the solution is described by the sum on the righthand side of this relation (see [1] or [3] for details).

So, if $\gamma \geq 0$ and $v > \gamma$, then a function $y : [\gamma - r, v] \to \mathbb{R}^n$ is called a solution of (1.1) on the interval $[\gamma, v]$ if it satisfies

$$y(t) = y(\gamma) + \int_\gamma^t f(y_s, s)\, ds + \sum_{\gamma \leq t_k < t} I_k(y(t_k))$$

for all $t \in (\gamma, v]$. The restriction $y_{|[\gamma - r,\gamma]}$ of the function y to the interval $[\gamma - r, \gamma]$ plays the role of the "starting condition (point)" for the solution y on $[\gamma, v]$.

The initial value problem (1.2) for the impulsive system is equivalent to the integral equation

$$\begin{cases} y\,(t) = y(a) + \displaystyle\int_a^t f\,(y_s, s)\,ds + \sum_{a \le t_k < t} I_k(y(t_k)) \ \text{ for } \ t > a, \\ y_a = \phi, \end{cases}$$

whenever the integral exists in some sense.

The integral occurring in our interpretation of solutions is the Lebesgue integral and therefore the requirement of Lebesgue integrability of the integrand is natural. This concept of a solution necessarily leads to its properties. Namely, a solution is the sum of an absolutely continuous function (given by the integral term) and a simple step function (given by the sum) with jumps at the points t_k.

For $y \in PC([-r, \infty), \mathbb{R}^n)$ we denote

$$\|y\|_{PC} = \sup_{s \in [-r,\infty)} |y(s)|.$$

In this way, the topology of locally uniform convergence on $PC([-r, \infty), \mathbb{R}^n)$ is given.

Given a compact interval $I = [a, b] \subset [-r, +\infty)$ and a function $\varphi \in PC(I, \mathbb{R}^n)$ we identify the function φ with its constant prolongation to the whole $[-r, +\infty)$, i. e. with the function

$$\widehat{\varphi}\,(t) = \begin{cases} \varphi(a), \ -r \le t \le a \\ \varphi(t), \ a \le t \le b \\ \varphi(b), \ b \le t \le \infty. \end{cases}$$

It is clear that the norm of $\varphi \in PC(I, \mathbb{R}^n)$ is the same as the norm $\|\widehat{\varphi}\|_{PC}$ of its extension $\widehat{\varphi} \in PC([-r, +\infty), \mathbb{R}^n)$.

Let $PC_1 \subset PC([-r, \infty), \mathbb{R}^n)$ be an open set with the following property: if y is an element of PC_1, $a \ge 0$ and $\bar{t} \in [a, \infty)$, then $\bar{y}$ given by

$$\bar{y}\,(t) = \begin{cases} y\,(t)\,, \ a - r \le t \le \bar{t} \\ y\,(\bar{t})\,, \ \bar{t} < t \le \infty \end{cases}$$

is also an element of PC_1. In particular, any open ball in $PC([-r, \infty), \mathbb{R}^n)$ has this property.

Let $H_1 \subset PC\left([-r,0],\mathbb{R}^n\right)$ be such that $\{y_t\,;\ t\in[a,+\infty), y\in PC_1\}\subset H_1$.

To be specific with the function f we will assume that $f\left(\psi,t\right): H_1\times[0,+\infty)\to\mathbb{R}^n$, for $y\in PC([-r,\infty),\mathbb{R}^n)$ the mapping $t\mapsto f\left(y_t,t\right)$ is locally Lebesgue integrable in $[0,+\infty)$ and the following conditions hold:

(A) there is a locally Lebesgue integrable function $M:[0,\infty)\to\mathbb{R}$ such that for all $x\in PC_1$ and all $u_1,u_2\in[0,+\infty)$, $u_1\leq u_2$,

$$\left|\int_{u_1}^{u_2} f\left(x_s,s\right)ds\right|\leq\int_{u_1}^{u_2} M\left(s\right)ds;$$

(B) there is a locally Lebesgue integrable function $L:[0,\infty)\to\mathbb{R}$ such that for all $x,y\in PC_1$ and all $u_1,u_2\in[0,+\infty)$, $u_1\leq u_2$,

$$\left|\int_{u_1}^{u_2}\left[f\left(x_s,s\right)-f\left(y_s,s\right)\right]ds\right|\leq\int_{u_1}^{u_2} L\left(s\right)\|x_s-y_s\|\,ds.$$

For the impulse operators $I_k:\mathbb{R}^n\to\mathbb{R}^n$, $k=1,2,\ldots$, we assume the following conditions:

(A') there is a constant $K_1>0$ such that for all $k=1,2,\ldots$ and all $x\in\mathbb{R}^n$,

$$|I_k(x)|\leq K_1;$$

(B') there is a constant $K_2>0$ such that for all $k=1,2,\ldots$ and all $x,y\in\mathbb{R}^n$,

$$|I_k(x)-I_k(y)|\leq K_2|x-y|.$$

In [1] it was proved that, under the conditions above, system (1.1) can be identified by a one-to-one correspondence with a system of generalized ODE's which takes values in a Banach space. Local existence and uniqueness of solutions are guaranteed.

In the present paper, we consider system (1.1) assuming $f(0,t)\equiv 0$ and conditions (A), (B), (A') and (B'). Our aim is to obtain results on Lyapunov uniform stability and uniform asymptotic stability of the trivial solution of (1.1). In order to do this, we consider the corresponding generalized ODE and the stability theory in this setting. Then, by means of Lyapunov functionals satisfying weak Krasovskiĭ-type conditions and the correspondence between generalized ODE's and impulsive retarded differential equations, we are able to get the desired results.

We organize the paper in the following manner. Section 2 is devoted to the basis of the theory of generalized ODEs. In Section 3, the concepts of variational stability and variational asymptotic stability for generalized ODEs are explored. In Section 4, we investigate the uniform Lyapunov stability and uniform asymptotic stability of impulsive retarded systems.

2. Generalized ordinary differential equations. A *tagged division* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection (τ_i, s_i), where $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$ is a division of $[a,b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, k$.

A *gauge* on a set $E \subset [a,b]$ is any function $\delta : E \to (0, +\infty)$.

Given a gauge δ on $[a,b]$, a tagged division $d = (\tau_i, s_i)$ is δ-*fine* if, for every i,

$$[s_{i-1}, s_i] \subset \{t \in [a,b]\,;\, |t - \tau_i| < \delta(\tau_i)\}.$$

We sometimes write $d = (\tau_i, [s_{i-1}, s_i])$ instead of $d = (\tau_i, s_i)$ when we want to specify the subintervals.

In the sequel we will use integration specified by the next definition.

Let X be a Banach space with the norm $\|\cdot\|_X$.

Definition 2.1. A function $U(\tau, t) : [a,b] \times [a,b] \to X$ is *Kurzweil integrable* if there is a unique element $I \in X$ such that given $\varepsilon > 0$, there is a gauge δ on $[a,b]$ such that for every δ-fine tagged division $d = (\tau_i, s_i)$ of $[a,b]$, we have

$$\|S(U,d) - I\|_X < \varepsilon,$$

where $S(U,d) = \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})]$. In this case, we write $I = \int_a^b DU(\tau, t)$.

Remark 2.2. Suppose the integral $\int_a^b DU(\tau,t)$ exists. Then we define

$$\int_b^a DU(\tau,t) = -\int_a^b DU(\tau,t).$$

This type of integration belongs to Jaroslav Kurzweil and it was described extensively in Chapter I of [3] for the case $X = \mathbb{R}^n$ (see Definition 1.2n in [3]). Checking the results concerning this integration in [3], it can be easily seen that the results presented there can be transferred without any changes to the case of X-valued functions $U(\tau,t) : [a,b] \times [a,b] \to X$. Let us mention a few of them.

The integral has the usual properties of linearity, additivity with respect to adjacent intervals, etc.

An important result, which will be used latter, concerns the integrability on subintervals. It is stated next (see Theorem 1.10 in [3]).

Lemma 2.3. *Let $U(\tau,t) : [a,b] \times [a,b] \to X$ be integrable over $[a,b]$. Then $\int_c^d DU(\tau,t)$ exists, for each subinterval $[c,d] \subset [a,b]$.*

The following result is an important Hake-type theorem based on the Saks-Henstock Lemma (see Theorem 1.14 in [3]).

Lemma 2.4. *Let a function $U : [a,b] \times [a,b] \to X$ be given such that U is integrable over $[a,c]$ for every $c \in [a,b)$ and let the limit*

$$\lim_{c \to b-} \left[\int_a^c DU(\tau,t) - U(b,c) + U(b,b) \right] = I \in X$$

exists. Then the function U is integrable over $[a,b]$ and

$$\int_a^b DU(\tau,t) = I.$$

Similarly, if the function U is integrable over $[c,b]$ for every $c \in (a,b]$ and the limit

$$\lim_{c \to a+} \left[\int_c^b DU(\tau,t) + U(a,c) - U(a,a) \right] = I \in X$$

exists, then the function U is integrable over $[a,b]$ and

$$\int_a^b DU(\tau,t) = I.$$

This leads to the following result (see Theorem 1.16 in [3]).

Lemma 2.5. *Let $U : [a,b] \times [a,b] \to X$ be integrable over $[a,b]$ and $c \in [a,b]$. Then*

$$\lim_{s \to c} \left[\int_a^s DU(\tau,t) - U(c,s) + U(c,c) \right] = \int_a^c DU(\tau,t).$$

Remark 2.6. Lemma 2.5 shows that the function defined by

$$s \in [a,b] \;\mapsto\; \int_a^s DU(\tau,t) \in X,$$

i.e. the *indefinite integral of U*, may not be continuous in general. The indefinite integral is continuous at a point $c \in [a,b]$ if and only if the function $U(c,\cdot) : [a,b] \to X$ is continuous at the point c. Notice that if $U : [a,b] \times [a,b] \to X$ is integrable over $[a,b]$, then by Lemma 2.3 the indefinite integral of the function U is well defined on the whole interval $[a,b]$.

Having the concept of Kurzweil integrability of a function $U : [a,b] \times [a,b] \to X$, we are able to define the notion of a generalized ordinary differential equation.

Let an open set $\Omega \subset X \times \mathbb{R}$ be given. Assume that $G : \Omega \to X$ is a given X-valued function $G(x,t)$ defined for $(x,t) \in \Omega$.

Definition 2.7. A function $x : [\alpha, \beta] \to X$ is called *a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DG(x, t) \tag{2.1}$$

on the interval $[\alpha, \beta] \subset \mathbb{R}$ if $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - x(\gamma) = \int_{\gamma}^{v} DG(x(\tau), t) \tag{2.2}$$

holds for every γ, $v \in [\alpha, \beta]$.

Given an initial condition $(z_0, t_0) \in \Omega$ the following definition of the solution of the initial value problem for the equation (2.1) will be used.

Definition 2.8. A function $x : [\alpha, \beta] \to X$ is a *solution of the generalized ordinary differential equation* (2.1) *with the initial condition* $x(t_0) = z_0$ *on the interval* $[\alpha, \beta] \subset \mathbb{R}$ if $t_0 \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - z_0 = \int_{t_0}^{v} DG(x(\tau), t) \tag{2.3}$$

holds for every $v \in [\alpha, \beta]$.

Let $-\infty < a < b < \infty$ and let us set

$$\Omega = O \times [a, b],$$

where $O \subset X$ is an open set (e.g. $O = B_c = \{x \in X;\ \|x\|_X < c\}$ for some $c > 0$).

Definition 2.9. A function $G : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$ if there exists a nondecreasing function $h : [a, b] \to \mathbb{R}$ such that

$$\|G(x, s_2) - G(x, s_1)\|_X \leq |h(s_2) - h(s_1)| \tag{2.4}$$

for all (x, s_2), $(x, s_1) \in \Omega$ and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\|_X \leq \|x - y\|_X |h(s_2) - h(s_1)| \tag{2.5}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$.

For functions $G \in \mathcal{F}(\Omega, h)$, we have the following information about solutions of the generalized differential equation (2.1). See Lemma 3.10 in [3] for a proof.

Lemma 2.10. *Assume that* $G : \Omega \to X$ *satisfies condition* (2.4). *If* $[\alpha, \beta] \subset [a, b]$ *and* $x : [\alpha, \beta] \to X$ *is a solution of* (2.1), *then the inequality*

$$\|x(s_1) - x(s_2)\|_X \leq |h(s_2) - h(s_1)|$$

holds for every $s_1, s_2 \in [\alpha, \beta]$.

Let $\operatorname{var}_\alpha^\beta(x)$ be the variation of a function $x : [\alpha, \beta] \to X$. Lemma 2.10 implies the following property of solutions of (2.1).

Corollary 2.11. *Assume that* $G : \Omega \to X$ *satisfies the condition* (2.4). *If* $[\alpha, \beta] \subset (a, b)$ *and* $x : [\alpha, \beta] \to X$ *is a solution of* (2.1), *then* x *is of bounded variation on* $[\alpha, \beta]$ *and*

$$\operatorname{var}_\alpha^\beta x \leq h(\beta) - h(\alpha) < +\infty.$$

Every point in $[\alpha, \beta]$ *at which the function* h *is continuous is a continuity point of the solution* $x : [\alpha, \beta] \to X$.

Moreover, we have the following result (see Lemma 3.12 in [3]).

Lemma 2.12. *If* $x : [\alpha, \beta] \to X$ *is a solution of* (2.1) *and* $G : \Omega \to X$ *satisfies condition* (2.4), *then*

$$x(\sigma+) - x(\sigma) = \lim_{s \to \sigma+} x(s) - x(\sigma) = G(x(\sigma), \sigma+) - G(x(\sigma), \sigma)$$

for $\sigma \in [\alpha, \beta)$ *and*

$$x(\sigma) - x(\sigma-) = x(\sigma) - \lim_{s \to \sigma-} x(s) = G(x(\sigma), \sigma) - G(x(\sigma), \sigma-)$$

for $\sigma \in (\alpha, \beta]$, *where*

$$G(x, \sigma+) = \lim_{s \to \sigma+} G(x, s), \quad \text{for } \sigma \in [\alpha, \beta)$$

and

$$G(x, \sigma-) = \lim_{s \to \sigma-} G(x, s), \quad \text{for } \sigma \in (\alpha, \beta].$$

Now we present a result on the existence of the integral involved in the definition of the solution of the generalized equation (2.1). This result is a particular case of Corollary 3.16 and of Proposition 3.6 in [3].

Lemma 2.13. *Let* $G \in \mathcal{F}(\Omega, h)$. *Suppose* $x : [a, b] \to X$ *is of bounded variation on* $[a, b]$ *and* $(x(s), s) \in \Omega$ *for every* $s \in [a, b]$. *Then the integral* $\int_a^b DG(x(\tau), t)$ *exists and the function* $s \mapsto \int_a^s DG(x(\tau), t) \in X$ *is of bounded variation.*

The next result concerns the existence of a solution of (2.1) (see [1], Theorem 2.15).

THEOREM 2.14 (EXISTENCE AND UNIQUENESS). *Let $G : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$, where the function h is left continuous (i.e. $h(t-) = h(t)$, $t \in (a, b]$). Then for every $(\widetilde{x}, t_0) \in \Omega$ such that for $\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)$ we have $(\widetilde{x}_+, t_0) \in \Omega$ and there exists a $\Delta > 0$ such that on the interval $[t_0, t_0 + \Delta]$ there exists a unique solution $x : [t_0, t_0 + \Delta] \to X$ of the generalized ordinary differential equation* (2.1) *for which $x(t_0) = \widetilde{x}$.*

REMARK 2.15. The assumption on the left continuity of the function h in Theorem 2.14 shows that the solutions of (2.1) are also left continuous (cf. Lemma 2.10). Given a solution x of (2.1), the limit $x(\sigma+)$ exists for every σ in the domain of x. This follows again by Lemma 2.10 and, by Lemma 2.12, we have the relation

$$x(\sigma+) = x(\sigma) + G(x(\sigma), \sigma+) - G(x(\sigma), \sigma)$$

which describes the discontinuity of the given solution from the right.

3. Some concepts of stability for GODE's. In this section we set $\Omega = B_c \times (a, b)$, where $B_c = \{y \in X; \|y\|_X < c\}$, $c > 0$, X is a Banach space and $(a, b) \subset [0, +\infty)$, $r > 0$. We also assume that $G \in \mathcal{F}(\Omega, h)$, where $h : [0, \infty) \to \mathbb{R}$ is a left continuous nondecreasing function, and $G(0, t) - G(0, s) = 0$, for $t, s \geq -r$. Then for every $[\gamma, v] \subset [0, +\infty)$, we have

$$\int_\gamma^v DG(0, t) = G(0, v) - G(0, \gamma) = 0$$

and, therefore, $x \equiv 0$ is a solution of the generalized equation (2.1) on $[0, +\infty)$. Note also that, by Lemma 2.10, every solution of (2.1) is continuous from the left.

If $G \in \mathcal{F}(\Omega, h)$ and $x : [\gamma, v] \to X$ is a solution of (2.1), where $[\gamma, v] \subset [0, +\infty)$, then x is of bounded variation on $[\gamma, v]$ by Corollary 2.11 and, of course, $x(t) \in B_c$ for $t \in [\gamma, v]$. Thus it is natural to measure the distance of two solutions using the variation norm given by $\|x(\gamma)\|_X + \mathrm{var}_\gamma^v x$.

The next stability concepts are based on the variation of the solutions around $x \equiv 0$.

DEFINITION 3.1. The solution $x \equiv 0$ of (2.1) is called *variationally stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $0 \leq \gamma < v < +\infty$ is a function of bounded variation on $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\|_X < \delta$$

and

$$\mathrm{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t) \right) < \delta,$$

then

$$\|\overline{x}(t)\|_X < \varepsilon, \quad t \in [\gamma, v].$$

The conditions in Definition 3.1 mean that the BV function $\overline{x}$ is close (in the "BV norm" : $\|\overline{x}(\gamma)\|_X + \mathrm{var}_\gamma^v(\overline{x}(s) - \int_\gamma^s DF(\overline{x}(\tau), t)))$ to the solution $x \equiv 0$ of (2.1).

Definition 3.2. The solution $x \equiv 0$ of (2.1) is called *variationally attracting* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $0 \leq \gamma < v < +\infty$, is a function of bounded variation on $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\|_X < \delta_0$$

and

$$\mathrm{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t) \right) < \rho,$$

then

$$\|\overline{x}(t)\|_X < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty), \ \gamma \geq 0.$$

Definition 3.3. The solution $x \equiv 0$ of (2.1) is called *variationally asymptotically stable* if it is variationally stable and variationally attracting.

To Defintions 3.1-3.3, it should be noted that if $\overline{x} : [\gamma, v] \to X$ is a solution of (2.1) then:

(a) $\overline{x}$ is of bounded variation on $[\gamma, v]$ and
(b) $\mathrm{var}_\gamma^v(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t)) = 0$.

Indeed, (a) follows by Corollary 2.11 and (b) is a consequence of the fact that for every $s \in [\gamma, v]$,

$$\overline{x}(s) - \overline{x}(\gamma) = \int_\gamma^s DF(\overline{x}(\tau), t)$$

and therefore

$$\mathrm{var}_\gamma^v(\overline{x}(s) - \int_\gamma^s DF(\overline{x}(\tau), t)) = 0.$$

The next lemma will be useful in the proofs of the Lyapunov-type theorems for the generalized equation (2.1). A proof of it for the case $X = \mathbb{R}^n$ can be found in [3], Lemma 10.12 and it holds also for a general Banach space.

Lemma 3.4. *Let $G \in \mathcal{F}(\Omega, h)$. Suppose $V : [0, +\infty) \times X \to \mathbb{R}$ is such that $V(\cdot, x) : [0, \infty) \to \mathbb{R}$ is left continuous on $(0, +\infty)$ for $x \in X$ and satisfies*

$$|V(t, z) - V(t, y)| \leq K\|z - y\|_X, \quad z, y \in X, \ t \in [0, +\infty), \tag{3.1}$$

where $K > 0$ is a constant. Suppose in addition that there is a function $\Phi : X \to \mathbb{R}$ such that for every solution $x : [a,b] \to X$, $[a,b] \subset [0,+\infty)$, of (2.1), we have

$$\limsup_{\eta\to 0_+} \frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} \leq \Phi(x(t)), \quad t \in [a,b]. \tag{3.2}$$

If $\overline{x} : [\gamma, v] \to X$, $0 \leq \gamma < v < +\infty$ is left continuous on $(\gamma, v]$ and of bounded variation on $[\gamma, v]$, then

$$V(v, \overline{x}(v)) - V(\gamma, \overline{x}(\gamma)) \leq K \operatorname{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t) \right) + M(v - \gamma), \tag{3.3}$$

where $M = \sup_{t\in[\gamma,v]} \Phi(\overline{x}(t))$.

Remark 3.5. In [2], variational stability for GODEs was studied in connection with stability with respect to perturbations and it was shown that these concepts are equivalent.

The next two results are Lyapunov-type theorems for equation (2.1) and they are borrowed from [3] (see [3], Theorems 10.13 and 10.14; see also [4]). We include their proofs here for the sake of self-containedness.

Theorem 3.6. *Let $V : [0,+\infty) \times \overline{B_\rho} \to \mathbb{R}$, where $\overline{B_\rho} = \{y \in X; \|y\|_X \leq \rho\}$, $0 < \rho < c$, be such that $V(\cdot, x) : [0, \infty) \to \mathbb{R}$ is left continuous on $(0, +\infty)$ for $x \in X$ and the following conditions hold:*

(i) $V(t, 0) = 0$, $t \in [0, +\infty)$;

(ii) there is a constant $K > 0$ such that

$$|V(t, z) - V(t, y)| \leq K\|z - y\|_X, \quad t \in [0, +\infty),\ z, y \in \overline{B_c}.$$

(iii) V is positive definite, that is, there is a function $b : [0, +\infty) \to \mathbb{R}$ of Hahn class such that

$$V(t, x) \geq b(\|x\|_X), \quad (t, x) \in [0, +\infty) \times \overline{B_c};$$

(iv) for all solutions $x(t)$ of (2.1),

$$\limsup_{\eta\to 0_+} \frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} \leq 0,$$

that is, the right derivative of V along every solution $x(t)$ of (2.1) is non-positive.

Then the trivial solution $x \equiv 0$ of (2.1) is variationally stable.

Proof. Let $\overline{x} : [\gamma, v] \to X$ be of bounded variation on $[\gamma, v]$ and left continuous on $]\gamma, v]$, $[\gamma, v] \subset [0, +\infty)$. By (iv) and Lemma 3.4, we have

(3.4)
$$V(t, \overline{x}(t)) \leq V(\gamma, \overline{x}(\gamma)) + K \operatorname{var}_\gamma^t \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), \overline{t}) \right), \quad t \in [\gamma, v].$$

If $\varepsilon > 0$ is given, then $b(\varepsilon) > 0$. Let $\delta(\varepsilon) > 0$ be such that $2K\delta(\varepsilon) < b(\varepsilon)$. If in addition $\|\overline{x}(\gamma)\|_X < \delta(\varepsilon)$ and

$$\operatorname{var}_\gamma^t \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t) \right) < \delta(\varepsilon),$$

then (3.4) implies

$$V(t, \overline{x}(t)) \leq 2K\delta(\varepsilon) < b(\varepsilon), \ t \in [\gamma, v] \tag{3.5}$$

since $V(\gamma, \overline{x}(\gamma)) \leq K\|\overline{x}(\gamma)\|_X$ by (i) and (ii).

On the other hand, if there exists $u \in [\gamma, v]$ such that $\|\overline{x}(u)\|_X \geq \varepsilon$, then (iii) implies

$$V(u, \overline{x}(u)) \geq b(\|\overline{x}(u)\|_X) \geq b(\varepsilon)$$

which contradicts (3.5). Hence $\|\overline{x}(t)\|_X < \varepsilon$ for all $t \in [\gamma, v]$ and the result follows. □

THEOREM 3.7. *Let* $V : [0, +\infty) \times \overline{B_\rho} \to \mathbb{R}$*, where* $\overline{B_\rho} = \{y \in X; \|y\|_X \leq \rho\}$*,* $0 < \rho < c$*, satisfies* (i) - (iii) *from Theorem 3.6. Suppose there is a continuous function* $\Phi : X \to \mathbb{R}$*, with* $\Phi(0) = 0$ *and* $\Phi(x) > 0$ *for* $x \neq 0$*, such that for every solution* $x : [\gamma, v] \to B_c$*,* $[\gamma, v] \subset [-r, +\infty)$*, of* (2.1)*, we have*

$$\limsup_{\eta \to 0_+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq -\Phi(x(t)), \ t \in [\gamma, v). \tag{3.6}$$

Then the trivial solution $x \equiv 0$ *of* (2.1) *is variationally asymptotically stable.*

Proof. By (3.6) it can be easily seen that (iv) from Theorem 3.6 holds. Thus Theorem 3.6 implies the trivial solution $x \equiv 0$ of (2.1) is variationally stable. Therefore it remains to prove that the solution $x \equiv 0$ of (2.1) is variationally attracting.

Since the solution $x \equiv 0$ of (2.1) is variationally stable, we have

(I) There is a $\delta_0 \in (0, \rho)$ such that if $\overline{x} : [\gamma, v] \to B_c$, $[\gamma, v] \subset [-r, +\infty)$, is a function of bounded variation on $[\gamma, v]$, left continuous on $(\gamma, v]$ and such that

$$\|\overline{x}(\gamma)\|_X < \delta_0$$

and

$$\mathrm{var}_{\gamma}^{v}\left(\overline{x}(s)-\int_{\gamma}^{s} DG(\overline{x}(\tau),t)\right)<\delta_0,$$

then

$$\|\overline{x}(t)\|_X < c, \quad t\in[\gamma,v].$$

(II) For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, $\delta < \varepsilon$ such that if $\overline{x} : [\overline{\gamma}, \overline{v}] \to B_c$, $[\overline{\gamma}, \overline{v}] \subset [-r, +\infty)$, is a function of bounded variation on $[\overline{\gamma}, \overline{v}]$, left continuous on $(\overline{\gamma}, \overline{v}]$ and such that

$$\|\overline{x}(\overline{\gamma})\|_X < \delta$$

and

$$\mathrm{var}_{\overline{\gamma}}^{\overline{v}}\left(\overline{x}(s)-\int_{\overline{\gamma}}^{s} DG(\overline{x}(\tau),t)\right)<\delta,$$

then

$$\|\overline{x}(t)\|_X < \varepsilon, \quad t\in[\overline{\gamma},\overline{v}].$$

Let $\lambda(\varepsilon) = \min\{\delta_0, \delta\} < \varepsilon$ and also

$$N = \sup\{-\Phi(y); \lambda(\varepsilon) \le \|y\|_X \le \varepsilon\} = -\inf\{\Phi(y); \lambda(\varepsilon) \le \|y\|_X \le \varepsilon\} < 0$$

and

$$T(\varepsilon) = \min\left\{v-\gamma, -K\frac{\delta_0+\lambda(\varepsilon)}{N}\right\} > 0. \tag{3.7}$$

Suppose $\overline{x} : [\gamma, v] \to B_c$, $[\gamma, v] \subset [-r, +\infty)$, is a function of bounded variation on $[\gamma, v]$, left continuous on $(\gamma, v]$ and such that

$$\|\overline{x}(\gamma)\|_X < \delta_0 \tag{3.8}$$

and

$$\mathrm{var}_{\gamma}^{v}\left(\overline{x}(s)-\int_{\gamma}^{s} DG(\overline{x}(\tau),t)\right)<\lambda(\varepsilon)<\delta. \tag{3.9}$$

We want to prove that

$$\|\overline{x}(t)\|_X < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T(\varepsilon), +\infty),\ \gamma \ge -r.$$

We assert that there exists $t^* \in [\gamma, \gamma + T(\varepsilon)]$ such that $\|\overline{x}(t^*)\|_X < \lambda(\varepsilon) \le \delta$. Indeed. Suppose the contrary, that is, $\|\overline{x}(s)\|_X \ge \lambda(\varepsilon)$, for every

$s \in [\gamma, \gamma + T(\varepsilon)]$. By Lemma 3.4, (ii) from Theorem 3.6, (3.9), (3.7) and (3.8), we have

$$\begin{aligned} V(\gamma+T(\varepsilon),\overline{x}(\gamma+T(\varepsilon))) &\leq V(\gamma,\overline{x}(\gamma)) \\ &+ K\,\mathrm{var}_{\gamma}^{\gamma+T(\varepsilon)}\left(\overline{x}(s) - \int_{\gamma}^{s} DG(\overline{x}(\tau),t)\right) + N\,T(\varepsilon) \\ &< K\|\overline{x}(\gamma)\|_X + K\,\lambda(\varepsilon) + N\left(-K\frac{\delta_0+\lambda(\varepsilon)}{N}\right) \\ &= K\|\overline{x}(\gamma)\|_X - K\delta_0 < 0. \end{aligned}$$

On the other hand, by (iii) from Theorem 3.6,

$$V(\gamma+T(\varepsilon),\overline{x}(\gamma+T(\varepsilon))) \geq b\left(\|\overline{x}(\gamma+T(\varepsilon))\|_X\right) \geq b(\lambda(\varepsilon)) > 0.$$

Hence we have a contradiction and the assertion holds. Therefore $\|\overline{x}(t)\|_X \leq \varepsilon$ for $t \in [t^*, v]$, since (II) holds for $\overline{\gamma} = t^*$ and $\overline{v} = v$. Also $\|\overline{x}(t)\|_X \leq \varepsilon$ for $t > \gamma + T(\varepsilon)$, since $t^* \in [\gamma, \gamma+T(\varepsilon)]$ and hence the solution $x \equiv 0$ of (2.1) is variationally attracting and we finished the proof. ∎

4. Stability of impulsive retarded systems. Now we turn our attention to impulsive retarded functional differential equations (1.1) as they are presented in the introductory Section 1 with the assumptions (A), (B), (A'), (B') satisfied. We want to establish stability results for these equations by means of generalized ODE's.

Definition 4.1. Assume that $t_0 \geq 0$ and $\sigma > 0$. A function $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ is called a *solution of* (1.1) *on the interval* $[t_0, t_0+\sigma]$ if $t \mapsto f(y_t, t)$ is Lebesgue integrable over $[t_0, t_0+\sigma]$, $y_t \in H_1$ for $t \in [t_0, t_0+\sigma]$ and

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\,ds + \sum_{t_0 \leq t_k < t} I_k(y(t_k))$$

holds for every $(t_0, t_0+\sigma]$.

It can be easily seen that for a solution $y : [t_0 - r, t_0+\sigma] \to \mathbb{R}^n$ of (1.1) on $[t_0, t_0+\sigma]$ we have

(i) $\dot{y}(t) = f(y_t, t)$ for almost every $t \in [t_0, t_0+\sigma]$

and

(ii) $y(t_k+) = y(t_k) + I_k(y(t_k))$ for all t_k such that $[t_0, t_0+\sigma)$.

Given $y \in PC_1$ and $t \in [-r, \infty)$, we define

$$F(y,t)(\vartheta) = \begin{cases} 0, & -r \leq \vartheta \leq 0 \ \text{or} \ -r \leq t \leq 0, \\ \int_a^{\vartheta} f(y_s, s)\,ds, & 0 \leq \vartheta \leq t < \infty, \\ \int_a^{t} f(y_s, s)\,ds, & 0 \leq t \leq \vartheta < \infty, \end{cases} \tag{4.1}$$

and

$$J(y,t)(\vartheta) = \sum_{l=1}^{\infty} H_{t_l}(t)H_{t_l}(\vartheta)I_l(y(t_l)) \tag{4.2}$$

for $\vartheta \in [-r,\infty)$, where H_{t_k} is the left continuous Heaviside function concentrated at t_k, that is,

$$H_{t_k}(t) = \begin{cases} 0 & \text{for} \quad t \le t_k \\ 1 & \text{for} \quad t_k < t. \end{cases}$$

Taking $F(y,t)$ from (4.1) and $J(y,t)$ from (4.2), define

$$G(y,t)(\vartheta) = F(y,t)(\vartheta) + J(y,t)(\vartheta) \tag{4.3}$$

for $y \in PC_1$, $t \in [0,\infty)$ and $\vartheta \in [-r,\infty)$. Clearly the values of the function $G(y,t)$ belong to $PC([a-r,\infty),\mathbb{R}^n)$, that is,

$$G : PC_1 \times [0,\infty) \to PC([-r,\infty),\mathbb{R}^n).$$

Moreover for $s_1, s_2 \in [0,\infty)$ and $x, y \in PC_1$ we have

$$\|G(x,s_2) - G(x,s_1)\| \le h(s_2) - h(s_1) \tag{4.4}$$

and

$$\|G(x,s_2) - G(x,s_1) - G(y,s_2) - G(y,s_1)\| \le \|x-y\|(h(s_2) - h(s_1)), \tag{4.5}$$

where

$$h(t) = \int_0^t [M(s) + L(s)]ds + \max(K_1,K_2)\sum_{k=1}^{\infty} H_{t_k}(t), \quad t \in [0,\infty).$$

is a nondecreasing real function which is continuous from the left at every point, continuous for all $t \ne t_l$ and $h(t_l+)$ exists for $l = 1, 2, \ldots$.

According to (4.4) and (4.5), it can be easily seen that the function G defined by (4.3) belongs to the class $\mathcal{F}(\Omega, h)$, where $\Omega = PC_1 \times [c,d]$ and $[c,d]$ is any compact subinterval of $[a,\infty)$.

For details, see [1].

Consider now the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG\,(x,t)\,. \tag{4.6}$$

Note that any solution of (4.6) on an interval $[\gamma, v]$ is a function $x(t)$ with values in $PC([-r,\infty),\mathbb{R}^n)$.

A solution x of (4.6) in the interval $[\gamma, v]$ has the following property (see Lemma 3.3 in [1]).

LEMMA 4.2. *Let $x(t)$ be a solution of* (4.6) *in the interval* $[\gamma, v]$. *Then if* $\sigma \in [\gamma, v]$ *we have*

(a) $\quad x(\sigma)(\vartheta) = x(\sigma)(\sigma), \quad \vartheta \geq \sigma, \ \vartheta \in [\gamma - r, v]$

and

(b) $\quad x(\sigma)(\vartheta) = x(\vartheta)(\vartheta), \quad \vartheta \leq \sigma, \ \vartheta \in [\gamma - r, v]$.

The next equivalence result gives a one-to-one relation between a solution of the impulsive retarded equation (1.1) and a solution of the generalized equation (4.6), with G given by (4.3). A proof of it can be found in [1], Theorems 3.4 and 3.5.

THEOREM 4.3 (EQUIVALENCE OF EQUATIONS).

(i) Consider equation (1.1), *where* $f : H_1 \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, $t \mapsto f(y_t, t)$ *is for* $y \in PC_1$ *Lebesgue integrable over* $[t_0, t_0 + \sigma]$ *and* (A), (B), (A′), (B′) *are fulfilled. Let* $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, $t_0 \geq 0$ *be a solution of the problem* (1.1) *in the interval* $[t_0, t_0 + \sigma]$. *Given* $t \in [t_0, t_0 + \sigma]$, *let*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), \ \vartheta \in [t_0 - r, t] \\ y(t), \ \vartheta \in [t, t_0 + \sigma]. \end{cases}$$

Then $x(t) \in PC([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *is a solution of* (4.6) *in* $[t_0, t_0 + \sigma]$.

(ii) Reciprocally, let $x : [t_0, t_0 + \sigma] \to PC([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *be a solution of* (4.6), *with G given by* (4.3), *in the interval* $[t_0, t_0 + \sigma]$. *For every* $\vartheta \in [t_0 - r, t_0 + \sigma]$, *define*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \leq \vartheta \leq t_0 \\ x(\vartheta)(\vartheta), & t_0 \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

Then $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ *is a solution of* (1.1) *in* $[t_0, t_0 + \sigma]$ *and*

$$y(\vartheta) = x(t_0 + \sigma)(\vartheta) \quad \text{for} \ \ \vartheta \in [t_0 - r, t_0 + \sigma].$$

Assume that $t_0 \in [0, \infty)$ and $\phi \in PC([-r, 0], \mathbb{R}^n)$ are given. Define the function $\widetilde{x} \in PC([-r, \infty))$ by

$$\widetilde{x}(\vartheta) = \begin{cases} \widetilde{x}(t_0 - r) = \phi(-r) \ \text{for} \ \vartheta \in [-r, t_0 - r], \\ \phi(\vartheta - t_0) \ \text{for} \ \vartheta \in [t_0 - r, t_0], \\ \widetilde{x}(t_0) = \phi(0) \ \text{for} \ \vartheta \in [t_0, \infty). \end{cases} \tag{4.7}$$

Looking at the initial value problem for (4.6), with $x(t_0) = \widetilde{x}$, the local existence and uniqueness Theorem 2.14, together with the equivalence result given in Theorem 4.3, can be used to obtain the following local existence and uniqueness result.

THEOREM 4.4. *Consider problem* (1.2). *If conditions* (A), (B), (A'), (B') *are fulfilled and if* $\widetilde{x} \in PC_1$ *from* (4.7) *is such that*

$$\widetilde{x} + H_{t_l}(\vartheta) I_l(\widetilde{x}(t_0)) \in PC_1 \tag{4.8}$$

whenever $t_0 = t_l$ *for some* $l = 1, 2, \dots$, *then there is a* $\Delta > 0$ *such that on the interval* $[t_0, t_0 + \Delta]$ *there exists a unique solution* $y : [t_0, t_0 + \Delta] \to \mathbb{R}^n$ *of problem* (1.2) *for which* $y_{t_0} = \phi$.

By Theorem 2.14 for $\widetilde{x} \in PC_1$ the relation

$$\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0) \in PC_1,$$

is needed. This condition assures that the solution of the initial value problem for the generalized ordinary differential equation (4.6) does not jump out of the set PC_1 immediately at the moment t_0. Notice that, in our situation where G is given by (4.3), we have $G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0) = 0$ if $t_0 \neq t_l$, $l = 1, 2, \dots$ and $[G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)](\vartheta) = H_{t_l}(\vartheta) I_l(\widetilde{x}(t_0))$ if $t_0 = t_l$ for some $l = 1, 2, \dots$. This implies condition (4.8) from Theorem 4.4.

By Theorem 4.3, we also have the opposite, i.e. there is a one-to-one correspondence between the solutions of the problem (1.2) and the solutions of the initial value problem for (4.6) with $x(t_0) = \widetilde{x}$.

In the next lines, we assume that

$$0 \in H_1,\ f(0, t) = 0 \ \text{ for all } \ t \text{ and } \ I_k(0) = 0,\ k = 1, 2, \dots .$$

This implies that the function $y \equiv 0$ is a solution of the system (1.1) on any interval contained in $[0, +\infty)$.

We also consider the set $E_c = \{\psi \in H_1;\ \|\psi\|_X < c\}$, $c > 0$ and a point $t_0 \in [0, \infty)$.

DEFINITION 4.5. The trivial solution $y \equiv 0$ of the system (1.1) is called *stable* if for every $\varepsilon > 0$ and $\gamma \geq 0$, there exists $\delta = \delta(\varepsilon, \gamma) > 0$ such that if $\phi \in E_c$ and $\overline{y} = \overline{y}(\gamma, \phi) : [\gamma - r, v] \to \mathbb{R}^n$, is a solution of (1.1) on $[\gamma, v]$ such that $\overline{y}_\gamma = \phi$ with

$$\|\phi\| < \delta,$$

then

$$\|\overline{y}_t(\gamma, \phi)\| < \varepsilon, \quad t \in [\gamma, v].$$

DEFINITION 4.6. The trivial solution $y \equiv 0$ of the system (1.1) is called *uniformly stable* if the number δ in Definition 4.5 does not depend on γ.

DEFINITION 4.7. The solution $y \equiv 0$ of (1.1) is called *uniformly asymptotically stable* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists a

$T = T(\varepsilon, \delta_0) \geq 0$ such that if $\phi \in E_c$, and $\overline{y} : [\gamma - r, v] \to \mathbb{R}^n$, is a solution of (1.1) on $[\gamma, v]$ such that $\overline{y}_\gamma = \phi$ and

$$\|\phi\| < \delta_0,$$

then

$$\|\overline{y}_t(\gamma, \phi)\| < \varepsilon, \qquad \text{for } \ t \in [\gamma, v] \cap [\gamma + T, +\infty).$$

We will apply Theorem 4.3 together with Theorems 3.6 and 3.7 to obtain stability results for problem (1.1).

Given a solution $y : [\gamma - r, v] \to \mathbb{R}^n$ of equation (1.1) on $[\gamma, v]$, $\gamma \geq 0$, $v > \gamma$, such that $y_t = \psi$ for some $t \in [\gamma, v]$, we have evidently $\psi \in PC([-r, 0], \mathbb{R}^n)$ and

$$\psi(\theta) = y_t\left(\theta\right) = y\left(t + \theta\right) = y(t) - \int_{t+\theta}^{t} f\left(y_s, s\right) ds - \sum_{t+\theta \leq t_k < t} I_k(y(t_k))$$

for $\theta \in [-r, 0]$. In this case, we write $y_{t+\eta} = y_{t+\eta}(t, \psi)$ for every $\eta \geq 0$ such that $t + \eta \in [\gamma, v]$.

Let x be the solution of the generalized equation (4.6) given for the solution $y : [\gamma - r, v] \to \mathbb{R}^n$ by Theorem 4.3(i) on the interval $[\gamma, v] \subset [0, +\infty)$, with initial condition $x(\gamma) = \sigma_\gamma$, where $\sigma_\gamma(\tau) = \phi(\tau - \gamma)$, $\gamma - r \leq \tau \leq \gamma$, and $\sigma_\gamma(\tau) = \phi(0)$ for $\tau \geq \gamma$. Then $x(t)(t + \theta) = y(t + \theta)$, for all $t \in [\gamma, v]$ and all $\theta \in [-r, 0]$ and hence $(x(t))_t = y_t$ for all $t \in [\gamma, v]$.

On the other hand, if $x(t) \in PC_1$ is a solution of (4.6) on $[\gamma, v] \subset [0, +\infty)$ with the initial condition $x(\gamma) = \sigma_\gamma$, where $\sigma_\gamma(\tau) = \phi(\tau - \gamma)$ for $\tau \in [\gamma - r, \gamma]$, and $\sigma_\gamma(\tau) = \phi(0)$ for $\tau \geq \gamma$, then Theorem 4.3(ii) implies that we can find a solution y of (1.1), with $t_0 = \gamma$, by means of x. Suppose $(x(t))_t = \psi$ for some $t \in [\gamma, v]$. In this case, we write $x_\psi(s)$ instead of $x(s)$ for $s \in [t, v]$ and we have $y_t = \psi$.

Assume that a functional

$$U : [0, +\infty) \times PC([-r, 0], \mathbb{R}^n) \to \mathbb{R}$$

is given.

U will play the role of a Lyapunov functional with respect to the impulsive retarded equation (1.1). We relate it to a functional $V : [0, +\infty) \times \overline{B_a} \to \mathbb{R}$ with respect to the generalized equation (4.6) by defining

$$V(t, x) = U(t, x_t) \tag{4.9}$$

for $t \geq 0$ and $x \in PC([-r, \infty), \mathbb{R}^n)$.

It is easy to show that

(I) if

$$U(t,0)=0 \text{ for } t\in[0,+\infty),$$

then

$$V(t,0)=0 \text{ for } t\in[0,+\infty).$$

and

(II) if there is a constant $K>0$ such that

$$|U(t,\psi)-U(t,\overline{\psi})|\le K\|\psi-\overline{\psi}\|,\quad t\in[-r,+\infty),\ \ \psi,\overline{\psi}\in\overline{E_\rho}$$

then

$$|V(t,x)-V(t,\overline{x})|\le K\|x-\overline{x}\|_{PC}$$

for $x,\overline{x}\in PC([-r,\infty),\mathbb{R}^n)$.

Indeed, for (II) we have

$$|V(t,x)-V(t,\overline{x})|=|U(t,x_t)-U(t,\overline{x}_t)|\le K\|x_t-\overline{x}_t)\|=$$
$$=K\sup_{\vartheta\in[-r,0]}|x(t+\vartheta)-\overline{x}(t+\vartheta)|\le K\sup_{s\in[-r,t]}|x(s)-\overline{x}(s)|\le$$
$$\le K\sup_{s\in[-r,\infty)}|x(s)-\overline{x}(s)|=K\|x-\overline{x}\|_{PC}.$$

For $U:[0,+\infty)\times PC([-r,0],\mathbb{R}^n)\to\mathbb{R}$, $t\in[0,+\infty)$ and a given function $y\in PC([t-r,t+\sigma],\mathbb{R}^n)$, $\sigma>0$, we have by (4.9)

$$\frac{U(t+\eta,y_{t+\eta})-U(t,y_t)}{\eta}=\frac{V(t+\eta,y)-V(t,y)}{\eta}$$

if $0<\eta\le\sigma$ and

$$\limsup_{\eta\to0_+}\frac{U(t+\eta,y_{t+\eta})-U(t,y_t)}{\eta}=\limsup_{\eta\to0_+}\frac{V(t+\eta,y)-V(t,y)}{\eta}\tag{4.10}$$

This notion of the right upper derivative of U along y is prepared for solutions y of (1.1) on some interval $[\gamma,v]$.

With the notation above, we will prove the next two results concerning stability of the trivial solution of equation (1.1) when (A), (B), (A'), (B') are fulfilled.

Theorem 4.8. *Suppose that (A), (B), (A'), (B') are fulfilled. Let U : $[0,+\infty)\times\overline{E_\rho}\to\mathbb{R}$ be such that $U(\cdot,\psi)$ is left continuous on $(0,+\infty)$, where $\overline{E_\rho}=\{\psi\in H_1;\|\psi\|\le\rho\}$, $0<\rho<c$. Let U fulfills the conditions*

(i) $U(t,0)=0$, $t\in[0,+\infty)$;

and

(ii) there is a constant $K > 0$ such that

$$|U(t,\psi) - U(t,\overline{\psi})| \leq K\|\psi - \overline{\psi}\|, \quad t \in [0,+\infty), \ \ \psi,\overline{\psi} \in \overline{E_\rho}.$$

Assume further that

(iii) there is a function $b : [0,+\infty) \to \mathbb{R}$ of Hahn class such that

$$b(\|y\|_{PC}) \leq U(t, y_t)$$

for any $y \in PC([-r,\infty),\mathbb{R}^n))$

and

(iv) for any solution y of (1.1) on $[\gamma, v]$ and $t \in [\gamma, v)$ the inequality

$$\limsup_{\eta \to 0_+} \frac{U(t+\eta, y_{t+\eta}) - U(t, y_t)}{\eta} \leq 0$$

holds.

Then the trivial solution $y \equiv 0$ of (1.1) is uniformly stable.

Proof. Notice that, since f satisfies (A) and (B) and I_k satisfies (A') and (B') for $k = 1, 2, \ldots$, the function G from the equation (4.6) belongs to $\mathcal{F}(\Omega, h)$.

Taking into account the function $V : [0,\infty) \times PC([-r,\infty),\mathbb{R}^n) \to \mathbb{R}$ given by (4.9) and its properties (I) and (II) presented above, we have

$$V(t, 0) = 0 \ \text{ for } \ t \in [0,+\infty).$$

and

$$|V(t,x) - V(t,\overline{x}| \leq K\|x - \overline{x}\|_{PC}$$

for $t \in [0,+\infty)$ and $x,\overline{x} \in PC([-r,\infty),\mathbb{R}^n)$.

By condition (iii) we have

$$b(\|y\|_{PC}) \leq U(t, y_t) = V(t, y)$$

for $y \in PC([-r,+\infty),\mathbb{R}^n)$.

This shows that the function V in general satisfies conditions (i), (ii) and (iii) from Theorem 3.6.

Assume that $x : [t_0, t_0+\sigma] \to PC([-r,+\infty),\mathbb{R}^n))$, $\sigma > 0$, is a solution of (4.6). Then for $t \in [t_0, t_0+\sigma)$ we have by definition (4.9)

$$V(t+\eta, x(t+\eta)) = U(t+\eta, x(t+\eta)_{t+\eta})$$

for all $\eta \geq 0$, provided $t + \eta \leq t_0 + \sigma$.

Note that by (ii) from Theorem 4.3, the function

$$y(\theta) = x(t+\eta)(\theta)$$

is a solution of the system (1.1) on $[t, t+\eta]$ and $x(t+\eta)_{t+\eta} = y_{t+\eta}$. Hence

$$U(t+\eta, x(t+\eta)_{t+\eta}) = U(t+\eta, y_{t+\eta})$$

for all $\eta \geq 0$. Therefore

$$\frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} = \frac{U(t+\eta, y_{t+\eta}) - U(t, y_t)}{\eta}$$

for all $\eta > 0$. By (iv) and (4.10) we get

$$\limsup_{\eta \to 0+} \frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta} =$$

$$= \limsup_{\eta \to 0+} \frac{U(t+\eta, y_{t+\eta}) - U(t, y_t)}{\eta} \leq 0$$

if $t \in [t_0, t_0 + \sigma)$. This implies condition (iv) from Theorem 3.6. The hypotheses of Theorem 3.6 are fulfilled and therefore the solution $x \equiv 0$ of the generalized equation (4.6) is variationally stable.

Thus, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_\rho$, $0 \leq \gamma < v < +\infty$, is a function of bounded variation on $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\| < \delta \tag{4.11}$$

and

$$\operatorname{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t) \right) < \delta, \tag{4.12}$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v]. \tag{4.13}$$

Let $\phi \in E_c$ and $\overline{y} : [\gamma - r, v] \to \mathbb{R}^n$ be a solution of (1.1) on $[\gamma, v]$ with $\overline{y}_\gamma = \phi$. Suppose

$$\|\phi\| < \delta. \tag{4.14}$$

We will prove that

$$\|\overline{y}_t(\gamma, \phi)\| < \varepsilon, \quad t \in [\gamma, v]. \tag{4.15}$$

Let

$$\overline{x}\,(t)\,(\tau) = \begin{cases} \overline{y}\,(\tau)\,, \ \tau \in [\gamma - r, t] \\ \overline{y}\,(t)\,, \ \tau \in [t, +\infty). \end{cases} \tag{4.16}$$

By Theorem 4.3 (i), $\overline{x}$ is on $[\gamma, v]$ a solution of the generalized equation (4.6) satisfying the initial condition $x(\gamma) = \sigma_\gamma$, where $\sigma_\gamma(\tau) = \phi(\tau)$, $\gamma - r \leq \tau \leq \gamma$, and $\sigma_\gamma(\tau) = \phi(0)$, $\tau \geq \gamma$. Also G in equation (4.6) belongs to $\mathcal{F}(\Omega, h)$ and this implies that $\overline{x}$ is of bounded variation on $[\gamma, v]$ (by conditions (A) and (A') and Corollary (2.11)).

By (4.16) and (4.14), we have

$$\|\overline{x}(\gamma)\| = \sup_\tau |\overline{x}(\gamma)(\tau)| = \sup_{\gamma - r \leq \tau \leq \gamma} |\overline{y}(\tau)| = \|\phi\| < \delta \tag{4.17}$$

and therefore (4.11) holds. Moreover

$$\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t) = \overline{x}(\gamma), \quad s \in [\gamma, v]. \tag{4.18}$$

Hence

$$\operatorname{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t) \right) = 0 < \delta$$

and by (4.12) we obtain (4.13). In particular, $\|\overline{x}(v)\| < \varepsilon$. But (4.16) implies that for any given $t \geq \gamma$,

$$\begin{aligned} \|y_t(\gamma, \phi)\| &= \sup_{-r \leq \theta \leq 0} |y(t + \theta)| \leq \sup_{\gamma - r \leq \tau \leq v} |y(\tau)| \\ &= \sup_{\gamma - r \leq \tau \leq v} |\overline{x}(v)(\tau)| = \sup_\tau |\overline{x}(v)(\tau)| \\ &= \|\overline{x}(v)\| < \varepsilon. \end{aligned} \tag{4.19}$$

Thus (4.15) follows and the proof is complete. □

The next theorem concerns the uniform asymptotic stability of the trivial solution of (1.1).

Theorem 4.9. *Consider equation* (1.1), *where* (A), (B), (A'), (B') *are fulfilled. Let* $U : [0, +\infty) \times \overline{E_\rho} \to \mathbb{R}$, $0 < \rho < c$, *satisfy* (i), (ii) *and* (iii) *from Theorem* 4.8. *Suppose there is a continuous function* $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *of Hahn class such that for every solution* $y : [\gamma - r, v] \to \overline{E_\rho}$ *of* (1.1) *on* $[\gamma, v] \subset [0, +\infty)$ *we have*

$$\limsup_{\eta \to 0_+} \frac{U(t + \eta, y_{t+\eta}) - U(t, y_t)}{\eta} \leq -\Lambda(\sup_{\tau \leq t} |y(\tau|) \tag{4.20}$$

for $t \in [\gamma, v)$. *Then the trivial solution* $y \equiv 0$ *of* (1.2) *is uniformly asymptotically stable.*

Proof. We assume the notation of the previous theorem.

Suppose $V : [0, +\infty) \times PC_1 \to \mathbb{R}$ is given by (4.9). Then the hypotheses of Theorem 3.6 are fulfilled.

Let $\Phi : PC \to \mathbb{R}$ be defined by

$$\Phi(z) = \Lambda(\|z\|_{PC})$$

for $z \in PC$. Then Φ is continuous, $\Phi(0) = 0$ and $\Phi(z) > 0$, for $z \neq 0$.

Assume that $x : [t_0, t_0 + \sigma] \to PC([-r, +\infty), \mathbb{R}^n))$, $\sigma > 0$ is a solution of (4.6). The procedure from the proof of the previous Theorem 4.8 can be repeated to state that by (ii) from Theorem 4.3 the function

$$y(\theta) = x(t + \eta)(\theta)$$

is a solution of the system (1.1) on $[t, t + \eta]$ while $x(t + \eta)_{t+\eta} = y_{t+\eta}$ for all $\eta > 0$, and by the assumption (4.20), we get

$$\limsup_{\eta \to 0+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} =$$

$$= \limsup_{\eta \to 0+} \frac{U(t + \eta, y_{t+\eta}) - U(t, y_t)}{\eta} \leq -\Lambda(\sup_{\tau \leq t} |y(\tau|).$$

However

$$\sup_{\tau \leq t} |y(\tau)| = \sup_{\tau \leq t} |x(t)(\tau)| = \|x(t)\|_{PC},$$

because for $\tau \leq$ we have (by Lemma 4.2 and (b) in Theorem 4.3)

$$y(\tau) = x(t + \eta)(\tau) = x(t)(\tau)$$

and for $\tau \geq t$ it is $x(t)(\tau) = x(t)(t)$ (a constant w.r.t τ).

Therefore

$$\limsup_{\eta \to 0+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq$$

$$\leq -\Lambda(\sup_{\tau \leq t} |y(\tau|) = -\Lambda(\|x(t)\|_{PC} = -\Psi(x)(t))$$

and the hypotheses of Theorem 3.7 are satisfied.

Hence $x \equiv 0$ is variationally asymptotically stable which means that if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $-r \leq \gamma < v < +\infty$, is a function of bounded variation on $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\| < \delta_0 \tag{4.21}$$

and

$$\operatorname{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DG(\overline{x}(\tau), t) \right) < \rho, \tag{4.22}$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty),\ \gamma \geq -r. \tag{4.23}$$

Assume there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists a $T = T(\varepsilon, \delta) \geq 0$ such that if $\phi \in E_c$, and $\overline{y} : [\gamma, v] \to \mathbb{R}^n$, with $[\gamma, v] \subset [-r, +\infty)$ and $[\gamma, v] \ni 0$, is solution of (1.2) ($t_0 = 0$) such that $\overline{y}_0 = \phi$ and

$$\|\phi\| < \delta_0. \tag{4.24}$$

We want to prove that

$$\|\overline{y}_t(0, \phi)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty). \tag{4.25}$$

□

Remark 4.10. Examining the proofs of Theorems 4.8 and 4.9, we can see that their first instances show the variational stability and asymptotic variational stability of the solution $x \equiv 0$ of the corresponding GODE. This means that the solution $y \equiv 0$ of (1.1) is, in the sense given in [2], variationally stable and asymptotically variationally stable respectively.

REFERENCES

[1] M. Federson; Š. Schwabik, Generalized ODEs approach to impulsive retarded differential equations, *Differential and Integral Equations* 19(11), (2006), 1201-1234.

[2] M. Federson; Š. Schwabik, Stability for retarded differential equations, *Ukr. Math. Journal* 60, (2008), 107-126.

[3] Š. Schwabik, *Generalized Ordinary Differential Equations*, World Scientific,Singapore, Series in Real Anal., vol. 5, 1992.

[4] Š. Schwabik, Variational stability for generalized ordinary differential equations, *Časopis Pěst. Mat.* 109(4), (1984), 389-420.