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Topological conjugation and asymptotic stability in impulsive semidynamical systems

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Abstract

We prove several results concerning topological conjugation of two impulsive semidynamical systems. In particular, we prove that the homeomorphism which defines the topological conjugation takes impulsive points to impulsive points; it also preserves limit sets, prolongational limit sets and properties as the minimality of positive impulsive orbits as well as stability and invariance with respect to the impulsive system. We also present the concepts of attraction and asymptotic stability in this setting and prove some related results.

Keywords: Impulsive semidynamical systems; Topological conjugation; Attractor; Stability

1. Introduction

Impulsive differential equations (IDE) are an important tool to describe the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. These equations are modelled by differential equations which describe the period of continuous variation of state and conditions which describe the discontinuities of first kind of the solution or of its derivatives at the moments of impulse.

The theory of IDE is an important area of investigation. In the present work we apply this theory to semidynamical systems. We start by presenting a summary of the basis of semidynamical

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systems with impulse effect. For details, see Refs. [2–9]. Then we define the concept of topological conjugation between two semidynamical systems with impulse and give some results on the structure of the phase space. We also deal with the concept of asymptotic stability for these systems.

2. Impulsive semidynamical systems

2.1. Basic definitions and terminology

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(\mathrm{X}, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function

$$\pi : \mathrm{X} \times \mathbb{R}_+ \to \mathrm{X}$$

fulfills the conditions:

(a) $\pi(x, 0) = x$, for all $x \in \mathrm{X}$,
(b) $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in \mathrm{X}$ and $t, s \in \mathbb{R}_+$,
(c) π is continuous.

We denote such system by $(\mathrm{X}, \pi, \mathbb{R}_+)$ or simply (X, π). Under the above conditions, when $\mathbb{R}_+$ is replaced by $\mathbb{R}$, the triple $(\mathrm{X}, \pi, \mathbb{R})$ is a *dynamical system*. For every $x \in \mathrm{X}$, we consider the continuous function $\pi_x : \mathbb{R}_+ \to \mathrm{X}$ given by $\pi_x(t) = \pi(x, t)$ and call the *trajectory* of x.

Let (X, π) be a semidynamical system. Given $x \in \mathrm{X}$, the *positive orbit* of x is given by

$$\mathrm{C}^+(x) = \{\pi(x, t)\colon\ t \in \mathbb{R}_+\}$$

which we also denote by $\pi^+(x)$. Given $x \in \mathrm{X}$ and $r \in \mathbb{R}_+$, we define

$$\mathrm{C}^+(x, r) = \{\pi(x, t)\colon\ 0 \leqslant t < r\}.$$

For $t \geqslant 0$ and $x \in \mathrm{X}$, we define

$$\mathrm{F}(x, t) = \{y\colon\ \pi(y, t) = x\}$$

and, for $\Delta \subset [0, +\infty)$ and $D \subset \mathrm{X}$, we define

$$\mathrm{F}(D, \Delta) = \bigcup\{\mathrm{F}(x, t)\colon\ x \in D \text{ and } t \in \Delta\}.$$

Then a point $x \in \mathrm{X}$ is called an *initial point*, if $\mathrm{F}(x, t) = \emptyset$ for $t > 0$.

Now we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ consists of a semidynamical system, (X, π), a non-empty closed subset M of X and a continuous function $\mathrm{I} : \mathrm{M} \to \mathrm{X}$ such that for every $x \in \mathrm{M}$, there exists $\varepsilon_x > 0$ such that

$$\mathrm{F}\big(x, (0, \varepsilon_x)\big) \cap \mathrm{M} = \emptyset \quad \text{and} \quad \pi\big(x, (0, \varepsilon_x)\big) \cap \mathrm{M} = \emptyset.$$

Notice that the points of M are isolated in every trajectory of the system (X, π). The set M is called the *impulsive set*, the function I is called the *impulse function* and we write $\mathrm{N} = \mathrm{I}(\mathrm{M})$. We also define

$$\mathrm{M}^+(x) = \big(\pi^+(x) \cap \mathrm{M}\big) \setminus \{x\}.$$

Lemma 2.1. *Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. Then for every $x \in \mathrm{X}$, there is a positive number s_1, $0 < s_1 \leqslant +\infty$, such that $\pi(x, t) \notin \mathrm{M}$, whenever $0 < t < s_1$, and $\pi(x, s_1) \in \mathrm{M}$ if $\mathrm{M}^+(x) \neq \emptyset$.*

Proof. When $\mathrm{M}^+(x) = \emptyset$, we can consider $s_1 = +\infty$ and we have $\pi(x,t) \notin \mathrm{M}$, for all $t > 0$. Now we suppose that $\mathrm{M}^+(x) \neq \emptyset$. Then there is $t_1 \in \mathbb{R}_+$ such that $\pi(x,t_1) \in \mathrm{M}$. Since $\pi_x : \mathbb{R}_+ \to \mathrm{X}$ is continuous and M is a non-empty closed set, then the compact subset $[0,t_1] \cap \pi_x^{-1}(\mathrm{M})$ of $\mathbb{R}_+$ admits a smallest element, $s_1 < +\infty$, satisfying the lemma. □

Let $(\mathrm{X},\pi;\mathrm{M},\mathrm{I})$ be an impulsive semidynamical system and $x \in \mathrm{X}$. By means of Lemma 2.1, it is possible to define a function $\phi : \mathrm{X} \to (0,+\infty]$ in the following manner: if $\mathrm{M}^+(x) = \emptyset$, then $\phi(x) = +\infty$, and if $\mathrm{M}^+(x) \neq \emptyset$, then $\phi(x)$ is the smallest number, denoted by s, such that $\pi(x,t) \notin \mathrm{M}$, for $t \in (0,s)$, and $\pi(x,s) \in \mathrm{M}$. This means that $\phi(x)$ is the least positive time for which the trajectory of x meets M. Then for each $x \in \mathrm{X}$, we call $\pi(x,\phi(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in $(\mathrm{X},\pi;\mathrm{M},\mathrm{I})$ is a function $\tilde{\pi}_x$ defined on the subset $[0,s)$ of $\mathbb{R}_+$ (s may be $+\infty$) in X. The description of such trajectory follows inductively as described in the following lines.

If $\mathrm{M}^+(x) = \emptyset$, then $\tilde{\pi}_x(t) = \pi(x,t)$, for all $t \in \mathbb{R}_+$, and $\phi(x) = +\infty$. However if $\mathrm{M}^+(x) \neq \emptyset$, it follows from Lemma 2.1 that there is a smallest positive number s_0 such that $\pi(x,s_0) = x_1 \in \mathrm{M}$ and $\pi(x,t) \notin \mathrm{M}$, for $0 < t < s_0$. Then we define $\tilde{\pi}_x$ on $[0,s_0]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x,t), & 0 \leqslant t < s_0, \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = \mathrm{I}(x_1)$ and $\phi(x) = s_0$.

Since $s_0 < +\infty$, the process now continues from x_1^+ on. If $\mathrm{M}^+(x_1^+) = \emptyset$, then we define $\tilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$, $s_0 \leqslant t < +\infty$, and $\phi(x_1^+) = +\infty$. When $\mathrm{M}^+(x_1^+) \neq \emptyset$, it follows again from Lemma 2.1 that there is a smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in \mathrm{M}$ and $\pi(x_1^+, t - s_0) \notin \mathrm{M}$, for $s_0 \leqslant t < s_0 + s_1$. Then we define $\tilde{\pi}_x$ on $[s_0, s_0 + s_1]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \leqslant t < s_0 + s_1, \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2^+ = \mathrm{I}(x_2)$ and $\phi(x_1^+) = s_1$.

Now we suppose that $\tilde{\pi}_x$ is defined on the interval $[t_{n-1}, t_n]$ and that $\tilde{\pi}_x(t_n) = x_n^+$, where $t_n = \sum_{i=0}^{n-1} s_i$. If $\mathrm{M}^+(x_n^+) = \emptyset$, then $\tilde{\pi}_x(t) = \pi(x_n^+, t - t_n)$, $t_n \leqslant t < +\infty$, and $\phi(x_n^+) = +\infty$. If $\mathrm{M}^+(x_n^+) \neq \emptyset$, then there exists $s_n \in \mathbb{R}_+$ such that $\pi(x_n^+, s_n) = x_{n+1} \in \mathrm{M}$ and $\pi(x_n^+, t - t_n) \notin \mathrm{M}$, for $t_n \leqslant t < t_{n+1}$. Besides

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x_n^+, t - t_n), & t_n \leqslant t < t_{n+1}, \\ x_{n+1}^+, & t = t_{n+1}, \end{cases}$$

where $x_{n+1}^+ = \mathrm{I}(x_{n+1})$ and $\phi(x_n^+) = s_n$. Notice that $\tilde{\pi}_x$ is defined on each interval $[t_n, t_{n+1}]$, where $t_{n+1} = \sum_{i=0}^{n} s_i$. Hence $\tilde{\pi}_x$ is defined on $[0, t_{n+1}]$.

The process above ends after a finite number of steps, whenever $\mathrm{M}^+(x_n^+) = \emptyset$ for some n. Or it continues infinitely, if $\mathrm{M}^+(x_n^+) \neq \emptyset$, $n = 1,2,3,\ldots,$ and if $\tilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

It worths noticing that given $x \in \mathrm{X}$, one of the three properties hold:

(i) $\mathrm{M}^+(x) = \emptyset$ and hence the trajectory of x has no discontinuities.
(ii) For some $n \geqslant 1$, each x_k^+, $k = 1,2,\ldots,n$, is defined and $\mathrm{M}^+(x_n^+) = \emptyset$. In this case, the trajectory of x has a finite number of discontinuities.
(iii) For all $k \geqslant 1$, x_k^+ is defined and $\mathrm{M}^+(x_k^+) \neq \emptyset$. In this case, the trajectory of x has infinite discontinuities.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Given $x \in X$, the *positive impulsive orbit* of x is defined by the set

$$\widetilde{C}^+(x) = \{\tilde{\pi}(x, t)\colon\ t \in \mathbb{R}_+\},$$

and we denote its closure in X by $\widetilde{K}^+(x)$.

Analogously to the non-impulsive case, we have the following properties.

Proposition 2.1. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system. If* $x \in X$*, then*

(i) $\tilde{\pi}(x, 0) = x$,
(ii) $\tilde{\pi}(\tilde{\pi}(x, t), s) = \tilde{\pi}(x, t + s)$*, with* $t, s \in [0, T(x))$ *such that* $t + s \in [0, T(x))$.

2.2. Semicontinuity and continuity of ϕ

In [6], the continuity of ϕ is discussed and the author assumes that ϕ is lower semicontinuous. But Ciesielski showed in [3] that this does not always hold.

The results of this section are borrowed from [3]. They are applied intrinsically in the proofs of the main theorems in the next section.

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x $(x \in X)$ is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ is a neighborhood of x;
(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leqslant \mu < \nu \leqslant 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*.

We include the complete proof of the next lemma.

Lemma 2.2. *Let* (X, π) *be a semidynamical system. If* S *is a* λ*-section through* x, $x \in X$*, and* $\mu \leqslant \lambda$*, then* S *is a* μ*-section through* x.

Proof. Consider the bar $L_\mu = F(L_\lambda, \lambda - \mu)$, where L_λ is a bar of the λ-tube. Notice that L_μ is closed, since π is continuous. Hence

(a) $F(L_\mu, \mu) = S$;
(b) $F(L_\mu, [0, 2\mu])$ is a neighborhood of x;
(c) $F(L_\mu, \sigma) \cap F(L_\mu, \nu) = \emptyset$, for $0 \leqslant \sigma < \nu \leqslant 2\mu$.

Indeed. We will prove each of these items.

(a) We have

$$\begin{aligned} x \in F(L_\mu, \mu) \quad &\Longleftrightarrow \quad \pi(x, \mu) \in L_\mu = F(L_\lambda, \lambda - \mu) \\ &\Longleftrightarrow \quad \pi\big(\pi(x, \mu), \lambda - \mu\big) \in L_\lambda \\ &\Longleftrightarrow \quad \pi(x, \lambda) \in L_\lambda \quad \Longleftrightarrow \quad x \in F(L_\lambda, \lambda) = S. \end{aligned}$$

(b) Since $F(L_\lambda, [0, 2\lambda])$ is a neighborhood of x, there is an open subset U_1 of X such that $x \in U_1 \subset F(L_\lambda, [0, 2\lambda])$. Let $T = F(L_\lambda, [0, \lambda - \mu] \cup [\lambda + \mu, 2\lambda])$. Notice that T is closed since given a sequence y_n in T, $y_n \to y$, there is a sequence $t_n \in [0, \lambda - \mu] \cup [\lambda + \mu, 2\lambda]$ such that $\pi(y_n, t_n) \in L_\lambda$. Since $[0, \lambda - \mu] \cup [\lambda + \mu, 2\lambda]$ is compact, we can assume, without loss of generality, that $t_n \to \tau$, $\tau \in [0, \lambda - \mu] \cup [\lambda + \mu, 2\lambda]$. Then the continuity of π implies $\pi(y_n, t_n) \to \pi(y, \tau)$. Because L_λ is closed, we have $\pi(y, \tau) \in L_\lambda$. Therefore $y \in T$. On the other hand, since $S \subset T^c$, where T^c is the complement of T in X, then there is an open set $U_2 \subset X$ containing x and such that $T \cap U_2 = \emptyset$. Thus $x \in U_1 \cap U_2 \subset F(L_\mu, [0, 2\mu])$.

(c) Suppose $y \in F(L_\mu, \sigma) \cap F(L_\mu, \nu)$, for $0 \leqslant \sigma < \nu \leqslant 2\mu$. Hence

$$\pi(y, \sigma) \in L_\mu = F(L_\lambda, \lambda - \mu) \quad \text{and} \quad \pi(y, \nu) \in L_\mu = F(L_\lambda, \lambda - \mu).$$

Therefore

$$\pi(y, \sigma + \lambda - \mu) \in L_\lambda \quad \text{and} \quad \pi(y, \nu + \lambda - \mu) \in L_\lambda,$$

which is a contradiction, since $0 \leqslant \sigma + \lambda - \mu < \nu + \lambda - \mu \leqslant 2\lambda$ and S is a λ-section. □

Let (X, π) be a semidynamical system. We now present the conditions TC and STC for a tube. Any tube $F(L, [0, 2\lambda])$ given by the section S through $x \in X$ such that

$$S \subset M \cap F\big(L, [0, 2\lambda]\big)$$

is called a *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition*, we write (TC), if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if

$$S = M \cap F\big(L, [0, 2\lambda]\big)$$

we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition*, we write (STC), if there exists an STC-tube $F(L, [0, 2\lambda])$ through x.

The following lemma is a consequence of these definitions.

Lemma 2.3. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system. Suppose there is a point* $x \in X$ *satisfying* (TC) (*respectively* (STC)) *with a* λ*-section* S. *Then given* $\eta < \lambda$, *the set* S *is an* η*-section with a* TC*-tube* (*respectively an* STC*-tube*).

The next result establishes a condition on a point of M so that the function ϕ is upper semicontinuous at it.

Theorem 2.1. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system. Suppose each point of the impulsive set* M *fulfills* (TC). *Then* ϕ *is upper semicontinuous.*

The following theorem states that if $x \notin M$, then ϕ is lower semicontinuous at x.

Theorem 2.2. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system. For each* $x \notin M$, *the function* ϕ *is lower semicontinuous at* x.

The next result says that ϕ is not lower semicontinuous at x, whenever $x \in M$ and x is not an initial point.

Theorem 2.3. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system. Suppose* $x \in M$ *and* x *is not an initial point. Then* ϕ *is not lower semicontinuous at* x.

The next result concerns the continuity of ϕ.

Theorem 2.4. *Consider the impulsive semidynamical system* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$. *Assume that no initial point belongs to the impulsive set* M *and that each element of* M *satisfies* (TC). *Then* ϕ *is continuous at* x *if and only if* $x \notin \mathrm{M}$.

Throughout this paper we consider that each element of M satisfies (TC).

2.3. Additional definitions

Let us consider the metric space X with metric ρ. By $B(x, \delta)$ we mean the open ball with center at $x \in \mathrm{X}$ and ratio δ. Let $B(A, \delta) = \{x \in \mathrm{X}\colon\ \rho_A(x) < \delta\}$, where $\rho_A(x) = \inf\{\rho(x, y)\colon y \in A\}$.

In what follows, $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is an impulsive semidynamical system and $x \in \mathrm{X}$.

We define the *limit set* of x in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ by

$$\widetilde{\mathrm{L}}^+(x) = \big\{y \in \mathrm{X}\colon\ \tilde{\pi}(x, t_n) \to y,\ \text{for some } t_n \to +\infty\big\}.$$

The *prolongational limit set* of x in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is given by

$$\widetilde{\mathrm{J}}^+(x) = \big\{y \in \mathrm{X}\colon \tilde{\pi}(x_n, t_n) \to y,\ \text{for } x_n \to x \text{ and } t_n \to +\infty\big\};$$

and the *prolongation set* of x in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is defined by

$$\widetilde{\mathrm{D}}^+(x) = \big\{y \in \mathrm{X}\colon \tilde{\pi}(x_n, t_n) \to y,\ \text{for } x_n \to x \text{ and } t_n \in [0, +\infty)\big\}.$$

We say that $\mathrm{C} \subset \mathrm{X}$ is *minimal* in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$, whenever $\mathrm{C} = \widetilde{\mathrm{K}}^+(x)$ for each $x \in \mathrm{C} \setminus \mathrm{M}$.

A point $x \in \mathrm{X}$ is called *stationary* with respect to $\tilde{\pi}$, if $\tilde{\pi}(x, t) = x$ for all $t \geqslant 0$, and it is called *periodic* with respect to $\tilde{\pi}$, if $\tilde{\pi}(x, t) = x$ for some $t > 0$ and x is not stationary.

Let $A \subset \mathrm{X}$. If $\tilde{\pi}^+(A) \subset A$, we say that A is $\tilde{\pi}$*-invariant*. If for every $\varepsilon > 0$ and every $x \in A$, there is $\delta > 0$ such that

$$\tilde{\pi}\big(B(x, \delta), [0, +\infty)\big) \subset B(A, \varepsilon),$$

then A is called $\tilde{\pi}$*-stable*. The set A is $\tilde{\pi}$*-orbitally stable* if for every neighborhood U of A, there is a positively $\tilde{\pi}$-invariant neighborhood V of A, $V \subset U$. If for all $x \in A$ and all $y \notin A$, there exist a neighborhood V of x and a neighborhood W of y such that $W \cap \tilde{\pi}(V, [0, +\infty)) = \emptyset$, we say that A is $\tilde{\pi}$*-stable according to Bhatia–Hajek*. Finally, the set A is I*-invariant* in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$, whenever $\mathrm{I}(x) \in A$ for all $x \in \mathrm{M} \cap A$, and A is I*-stable* if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\mathrm{I}\big(\mathrm{M} \cap B(A, \delta)\big) \subset B(A, \varepsilon).$$

3. The main results

3.1. Topological conjugation

The qualitative study of an impulsive differential equation consists of the geometric description of its phase space. It is natural to ask when two phase spaces have the same description. This depends on an equivalence relation between impulsive equations. An equivalence relation that expresses the geometric structure of the orbits is a topological conjugation.

Let X and Y be metric spaces with metrics ρ_{X} and ρ_{Y}, respectively. Let $(\mathrm{X}, \pi; \mathrm{M}_{\mathrm{X}}, \mathrm{I}_{\mathrm{X}})$ and $(\mathrm{Y}, \psi; \mathrm{M}_{\mathrm{Y}}, \mathrm{I}_{\mathrm{Y}})$ be impulsive semidynamical systems. We say that X and Y are *topologically* *conjugate*, if there exists a homeomorphism $h: \mathrm{X} \to \mathrm{Y}$ which takes orbits of X to orbits of Y and preserves orientation, that is, $h(\widetilde{\mathrm{C}}^+_{\mathrm{X}}(p)) = \widetilde{\mathrm{C}}^+_{\mathrm{Y}}(h(p))$, with $h(\tilde{\pi}(p,t)) = \tilde{\psi}(h(p),t)$ for every $t \in T_{\mathrm{X}}(p) = T_{\mathrm{Y}}(h(p))$.

By the following proposition, it follows that the homeomorphism h takes impulsive points to impulsive points.

Proposition 3.1. *Let* $(\mathrm{X}, \pi; \mathrm{M_X}, \mathrm{I_X})$ *and* $(\mathrm{Y}, \psi; \mathrm{M_Y}, \mathrm{I_Y})$ *be impulsive semidynamical systems. Let* X *and* Y *be topologically conjugate by the homeomorphism h. Then*

$$\phi_{\mathrm{X}}(p) = \phi_{\mathrm{Y}}\big(h(p)\big), \quad \textit{for all } p \in \mathrm{X}.$$

Proof. Given $p \in \mathrm{X}$, we have

$$\tilde{\pi}(p,t) = \begin{cases} \pi(p,t), & 0 \leqslant t < \phi_{\mathrm{X}}(p), \\ p_1^+, & t = \phi_{\mathrm{X}}(p). \end{cases}$$

Then

$$h\big(\tilde{\pi}(p,t)\big) = \begin{cases} h(\pi(p,t)), & 0 \leqslant t < \phi_{\mathrm{X}}(p), \\ h(p_1^+), & t = \phi_{\mathrm{X}}(p). \end{cases}$$

Hence

$$\tilde{\psi}\big(h(p),t\big) = \begin{cases} \psi(h(p),t), & 0 \leqslant t < \phi_{\mathrm{X}}(p), \\ h(p_1^+), & t = \phi_{\mathrm{X}}(p). \end{cases}$$

If $\rho_{\mathrm{X}}(\pi(p,\phi_{\mathrm{X}}(p)), p_1^+) > 0$, since h is a homeomorphism, it follows that

$$\rho_{\mathrm{Y}}\big(\psi\big(h(p),\phi_{\mathrm{X}}(p)\big), h\big(p_1^+\big)\big) > 0.$$

Therefore $\phi_{\mathrm{Y}}(h(p)) = \phi_{\mathrm{X}}(p)$. □

In what follows, we assume that $(\mathrm{X}, \pi; \mathrm{M_X}, \mathrm{I_X})$ and $(\mathrm{Y}, \psi; \mathrm{M_Y}, \mathrm{I_Y})$ are impulsive semidynamical systems which are topologically conjugate by the homeomorphism h.

Proposition 3.2. *The following properties hold*:

(i) $h(\widetilde{\mathrm{L}}^+(p)) = \widetilde{\mathrm{L}}^+(h(p))$, *for all* $p \in \mathrm{X}$;
(ii) $h(\widetilde{\mathrm{J}}^+(p)) = \widetilde{\mathrm{J}}^+(h(p))$, *for all* $p \in \mathrm{X}$;
(iii) $h(\widetilde{\mathrm{D}}^+(p)) = \widetilde{\mathrm{D}}^+(h(p))$, *for all* $p \in \mathrm{X}$.

Proof. We prove (i). The proofs of (ii) and (iii) follow analogously.

Let $x \in h(\widetilde{\mathrm{L}}^+(p))$. Then there is $y \in \widetilde{\mathrm{L}}^+(p)$ such that $h(y) = x$. Thus there exists a sequence $\{t_n\} \subset \mathbb{R}_+$ such that

$$\tilde{\pi}(p,t_n) \to y, \quad \text{as } t_n \to +\infty.$$

Since h is continuous, we have

$$h\big(\tilde{\pi}(p,t_n)\big) \to h(y), \quad \text{as } t_n \to +\infty.$$

But $h(\tilde{\pi}(p,t_n)) = \tilde{\psi}(h(p),t_n)$, $t_n \in \mathbb{R}_+$. Therefore

$$\tilde{\psi}\big(h(p),t_n\big) \to x, \quad \text{as } t_n \to +\infty$$

and hence $x \in \widetilde{\mathrm{L}}^+(h(p))$.

Now we suppose $x \in \widetilde{\mathrm{L}}^+(h(p))$. Then there is a sequence $\{t_n\} \subset \mathbb{R}_+$ such that

$$\tilde{\psi}\big(h(p), t_n\big) \to x, \quad \text{as } t_n \to +\infty.$$

Since $h(\tilde{\pi}(p, t_n)) = \tilde{\psi}(h(p), t_n)$, $t_n \in \mathbb{R}_+$, we have

$$h\big(\tilde{\pi}(p, t_n)\big) \to x, \quad \text{as } t_n \to +\infty.$$

But h is a homeomorphism. Therefore there exists h^{-1} continuous and

$$\tilde{\pi}(p, t_n) \to h^{-1}(x), \quad \text{as } t_n \to +\infty.$$

Thus $h^{-1}(x) \in \widetilde{\mathrm{L}}^+(p)$ and hence $x \in h(\widetilde{\mathrm{L}}^+(p))$. The proof is then complete. □

Propositions 3.3 and 3.4 below show that the invariance is preserved by the homeomorphism h.

Proposition 3.3. *If* $\mathrm{A} \subset \mathrm{X}$ *is* $\tilde{\pi}$*-invariant, then* $h(\mathrm{A})$ *is* $\tilde{\psi}$*-invariant.*

Proof. For each $x \in \mathrm{A}$, we have $\tilde{\pi}(x, t) \subset \mathrm{A}$, for all $t \in \mathbb{R}_+$. Consider $p \in h(\mathrm{A})$. Then there exists $y \in \mathrm{A}$ such that $h(y) = p$. Thus $\tilde{\pi}(y, t) \subset \mathrm{A}$, for all $t \in \mathbb{R}_+$ and hence, $h(\tilde{\pi}(y, t)) \subset h(\mathrm{A})$, for all $t \in \mathbb{R}_+$. But $h(\tilde{\pi}(y, t)) = \tilde{\psi}(h(y), t)$. Therefore $\tilde{\psi}(h(y), t) \subset h(\mathrm{A})$ and the result follows. □

Proposition 3.4. *Let* $\mathrm{A} \subset \mathrm{X}$ *be such that* $\mathrm{A} \cap \mathrm{M_X}$ *is* $\mathrm{I_X}$*-invariant. Then* $h(\mathrm{A}) \cap \mathrm{M_Y}$ *is* $\mathrm{I_Y}$*-invariant.*

Proof. Let $y \in h(\mathrm{A}) \cap \mathrm{M_Y}$. Thus there exists $x \in \mathrm{A} \cap \mathrm{M_X}$ such that $h(x) = y$. By the hypothesis, $\mathrm{I_X}(x) \in \mathrm{A} \cap \mathrm{M_X}$. But $\mathrm{I_X}(x) = x_1^+ = \tilde{\pi}(x_1^+, 0)$. Hence

$$\mathrm{I_Y}(y) = \mathrm{I_Y}\big(h(x)\big) = \tilde{\psi}\big(h\big(x_1^+\big), 0\big) = h\big(\tilde{\pi}\big(x_1^+, 0\big)\big) = h\big(\mathrm{I_X}(x)\big) \in h(\mathrm{A}) \cap \mathrm{M_Y}$$

and the result follows. □

The next proposition says that the stability is also preserved by the homeomorphism h.

Proposition 3.5. *Let* $A \subset \mathrm{X}$. *We have*

(i) *If* A *is compact and* $\tilde{\pi}$*-stable, then* $h(A)$ *is* $\tilde{\psi}$*-stable.*
(ii) *If* A *is orbitally* $\tilde{\pi}$*-stable, then* $h(A)$ *is orbitally* $\tilde{\psi}$*-stable.*
(iii) *If* A *is* $\tilde{\pi}$*-stable according to Bhatia–Hajek, then* $h(A)$ *is* $\tilde{\psi}$*-stable according to Bhatia–Hajek.*

Proof. (i) Suppose there exists $\varepsilon > 0$ such that for every $\delta > 0$, $\tilde{\psi}(B(h(x), \delta))$ is not contained in $B(h(A), \varepsilon)$. Then there exists $h(\bar{x}) \in B(h(x), \delta)$ such that $\tilde{\psi}(h(\bar{x}), T) \notin B(h(A), \varepsilon)$, for some $T > 0$, that is, $h(\tilde{\pi}(\bar{x}, T)) \notin B(h(A), \varepsilon)$. Hence $\tilde{\pi}(\bar{x}, T) \notin h^{-1}(B(h(A), \varepsilon))$. Besides, $h^{-1}(B(h(A), \varepsilon))$ is a neighborhood of A. By the compactness of A there exists $\eta > 0$ such that $A \subset B(A, \eta) \subset h^{-1}(B(h(A), \varepsilon))$. By the $\tilde{\pi}$-stability of A, there exists $\beta > 0$ such that $\tilde{\pi}(B(\bar{x}, \beta)) \subset B(A, \eta)$, which is a contradiction, since $\tilde{\pi}(\bar{x}, T) \notin h^{-1}(B(h(A), \varepsilon))$. Hence the result follows.

(ii) For every neighborhood U of A, there is a $\tilde{\pi}$-invariant neighborhood V of A such that $V \subset U$. Then $h(A) \subset h(V) \subset h(U)$. Since h is a homeomorphism, $h(U)$ and $h(V)$ are neighborhoods of $h(A)$. Therefore, by Proposition 3.3, $h(V)$ is $\tilde{\psi}$-invariant.

(iii) For every $x \in A$ and every $y \notin A$, there are neighborhoods V of x and W of y such that $W \cap \tilde{\pi}(V, [0, +\infty)) = \emptyset$. Since h is injective, we have $h(W) \cap h(\tilde{\pi}(V, [0, +\infty))) = \emptyset$, that is, $h(W) \cap \tilde{\psi}(h(V), [0, +\infty)) = \emptyset$, where $h(x) \in h(A)$, $h(y) \notin h(A)$, and $h(V)$ and $h(W)$ are neighborhoods of $h(x)$ and $h(y)$, respectively. □

Proposition 3.6. *If $\widetilde{\mathrm{C}}^+_{\mathrm{X}}(p)$ is minimal, then $\widetilde{\mathrm{C}}^+_{\mathrm{Y}}(h(p))$ is also minimal.*

Proof. We need to prove that $\widetilde{\mathrm{C}}^+_{\mathrm{Y}}(h(p)) = \widetilde{\mathrm{K}}^+_{\mathrm{Y}}(y)$, for every $y \in \widetilde{\mathrm{C}}^+_{\mathrm{Y}}(h(p)) \setminus \mathrm{M}_{\mathrm{Y}}$. Suppose $w \in \widetilde{\mathrm{C}}^+_{\mathrm{Y}}(h(p))$. Then $w = \tilde{\psi}(h(p), T)$, for some $T \geqslant 0$. Hence $w = h(\tilde{\pi}(p, T))$. Since $\widetilde{\mathrm{C}}^+_{\mathrm{X}}(p)$ is minimal, we have $h^{-1}(w) \in \widetilde{\mathrm{K}}^+_{\mathrm{X}}(z)$, for all $z \in \widetilde{\mathrm{C}}^+_{\mathrm{X}}(p) \setminus \mathrm{M}_{\mathrm{X}}$. Thus for all $z \in \widetilde{\mathrm{C}}^+_{\mathrm{X}}(p) \setminus \mathrm{M}_{\mathrm{X}}$ there exists a sequence $\{t_n\} \subset \mathbb{R}_+$ such that

$$\tilde{\pi}(z, t_n) \to \tilde{\pi}(p, T).$$

Because h is continuous, we have

$$h\big(\tilde{\pi}(z, t_n)\big) \to h\big(\tilde{\pi}(p, T)\big),$$

that is,

$$\tilde{\psi}\big(h(z), t_n\big) \to \tilde{\psi}\big(h(p), T\big).$$

Therefore $w \in \widetilde{\mathrm{K}}^+_{\mathrm{Y}}(y)$, for every $y \in \widetilde{\mathrm{C}}^+_{\mathrm{Y}}(h(p)) \setminus \mathrm{M}_{\mathrm{Y}}$.

Now we suppose $y \in \widetilde{\mathrm{K}}^+_{\mathrm{Y}}(z)$, for every $z \in \widetilde{\mathrm{C}}^+_{\mathrm{Y}}(h(p)) \setminus \mathrm{M}_{\mathrm{Y}}$. Since h is a homeomorphism, we have $\widetilde{\mathrm{K}}^+_{\mathrm{Y}}(z) = h(\widetilde{\mathrm{K}}^+_{\mathrm{X}}(h^{-1}(z)))$. Thus $h^{-1}(y) \in \widetilde{\mathrm{K}}^+_{\mathrm{X}}(h^{-1}(z))$. But $\widetilde{\mathrm{K}}^+_{\mathrm{X}}(h^{-1}(z)) = \widetilde{\mathrm{C}}^+_{\mathrm{X}}(p)$. Therefore $y \in h(\widetilde{\mathrm{C}}^+_{\mathrm{X}}(p))$, that is, $y \in \widetilde{\mathrm{C}}^+_{\mathrm{Y}}(h(p))$ and the proof is complete. □

3.2. Asymptotic stability

In [10], the asymptotic stability is studied for non-impulsive semidynamical systems. Here we introduce this concept for the impulsive case and verify whether some properties still hold.

In what follows, $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is an impulsive semidynamical system.

Let $\mathrm{H} \subset \mathrm{X}$. We define the sets

$$\widetilde{\mathrm{P}}^+_{\mathrm{W}}(\mathrm{H}) = \big\{x \in \mathrm{X}\colon \text{ for every neighborhood } U \text{ of H, there is a sequence } \{t_n\} \subset \mathbb{R}_+,\ t_n \to +\infty \text{ such that } \tilde{\pi}(x, t_n) \in U\big\}$$

$$\widetilde{\mathrm{P}}^+(\mathrm{H}) = \big\{x \in \mathrm{X}\colon \text{ for every neighborhood } U \text{ of H, there is } \tau \in \mathbb{R}_+ \text{ such that } \tilde{\pi}(x, [\tau, +\infty)) \subset U\big\}.$$

The set $\widetilde{\mathrm{P}}^+_{\mathrm{W}}(\mathrm{H})$ is called a *region of weak attraction* of H according to $\tilde{\pi}$ and the set $\widetilde{\mathrm{P}}^+(\mathrm{H})$ is called a *region of attraction* of H according to $\tilde{\pi}$. If $x \in \widetilde{\mathrm{P}}^+_{\mathrm{W}}(\mathrm{H})$ or $x \in \widetilde{\mathrm{P}}^+(\mathrm{H})$, then we say that x is *$\tilde{\pi}$-weakly attracted* or *$\tilde{\pi}$-attracted* to H, respectively.

Lemma 3.1. *For any set* $\mathrm{H} \subset \mathrm{X}$*, we have*

(i) $\widetilde{\mathrm{P}}^+(\mathrm{H}) \subset \widetilde{\mathrm{P}}^+_{\mathrm{W}}(\mathrm{H})$;
(ii) $\widetilde{\mathrm{P}}^+(\mathrm{H})$ *and* $\widetilde{\mathrm{P}}^+_{\mathrm{W}}(\mathrm{H})$ *are $\tilde{\pi}$-invariant.*

Proof. (i) follows immediately.

(ii) We will show that $\widetilde{P}^+(H)$ is $\tilde{\pi}$-invariant. The $\tilde{\pi}$-invariance of $\widetilde{P}^+_W(H)$ follows analogously.

Consider $y \in \widetilde{P}^+(H)$. Let U be an arbitrary neighborhood of H. Thus there exists $\tau \in \mathbb{R}_+$ such that $\tilde{\pi}(y, [\tau, +\infty)) \subset U$.

Now, consider $z = \tilde{\pi}(y, \lambda)$, $\lambda \in \mathbb{R}_+$. Then for every $t \in [\tau, +\infty)$, we have

$$\tilde{\pi}(z, t) = \tilde{\pi}\big(\tilde{\pi}(y, \lambda), t\big) = \tilde{\pi}(y, t + \lambda) \in U.$$

Hence $\widetilde{C}^+(y) \subset \widetilde{P}^+(H)$. □

Lemma 3.2. *Given* $H \subset X$ *and* $x \in X$*, suppose there is a sequence* $\{t_n\} \subset \mathbb{R}_+$*, with* $t_n \to +\infty$*, such that either* $\tilde{\pi}(x, t_n) \in H$ *or* $H \cap \widetilde{L}^+(x) \neq \emptyset$*. Then* $x \in \widetilde{P}^+_W(H)$.

Proof. Suppose $x \notin \widetilde{P}^+_W(H)$. Thus there are an open neighborhood U of H and $\tau \in \mathbb{R}_+$ such that $\tilde{\pi}(x, [\tau, +\infty)) \subset X \setminus U$. Since $X \setminus U$ is closed, then $\widetilde{K}^+(\tilde{\pi}(x, \tau)) \subset X \setminus U$. Thus $H \cap \widetilde{L}^+(x) = \emptyset$ and, for every $t_n \to +\infty$, we have $\tilde{\pi}(x, t_n) \notin H$ which contradicts the hypothesis. □

Remark 3.1. If in Lemma 3.2, we have in addition that $H \subset X$ is an open subset of X, the boundary of H is the impulsive set $M \neq \emptyset$ and $I(M) \subset H$, then the converse holds.

Proof. Let $x \in \widetilde{P}^+_W(H)$. Suppose there is no sequence $\{t_n\}$ in $\mathbb{R}_+$, $t_n \to +\infty$, with $\tilde{\pi}(x, t_n) \in H$. Thus there exists $\tau \in \mathbb{R}_+$, with $\widetilde{C}^+(\tilde{\pi}(x, \tau)) \subset X \setminus H$. If $H \cap \widetilde{L}^+(x) = \emptyset$, then

$$\widetilde{K}^+\big(\tilde{\pi}(x, \tau)\big) = \widetilde{C}^+\big(\tilde{\pi}(x, \tau)\big) \cup \widetilde{L}^+\big(\tilde{\pi}(x, \tau)\big) = \widetilde{C}^+\big(\tilde{\pi}(x, \tau)\big) \cup \widetilde{L}^+(x) \subset X \setminus H,$$

for all $x \in X$, $\widetilde{K}^+(x) = \widetilde{C}^+(x) \cup \widetilde{L}^+(x)$ (see [9, Lemma 2.10]). Since $\widetilde{K}^+(\tilde{\pi}(x, \tau))$ is closed, $X \setminus \widetilde{K}^+(\tilde{\pi}(x, \tau))$ is a neighborhood of H. As a consequence, we do not have $\tilde{\pi}(x, \tau) \in \widetilde{P}^+_W(H)$ which implies $x \notin \widetilde{P}^+_W(H)$ and we have a contradiction. □

A set $H \subset X$ is called a *weak* $\tilde{\pi}$*-attractor*, if $\widetilde{P}^+_W(H)$ is a neighborhood of H, and it is called a $\tilde{\pi}$*-attractor*, if $\widetilde{P}^+(H)$ is a neighborhood of H.

Proposition 3.7. *If* $H \subset X$ *is a* $\tilde{\pi}$*-attractor, then* $\widetilde{P}^+(H) = \widetilde{P}^+_W(H)$.

Proof. By Lemma 3.1, it is enough to prove that $\widetilde{P}^+_W(H) \subset \widetilde{P}^+(H)$. Let $x \in \widetilde{P}^+_W(H)$. Since $\widetilde{P}^+(H)$ is a neighborhood of H, there exists $\tau \in \mathbb{R}_+$ such that $\tilde{\pi}(x, \tau) \in \widetilde{P}^+(H)$. But $\widetilde{P}^+(H)$ is $\tilde{\pi}$-invariant. Hence $x \in \widetilde{P}^+(H)$. □

A set $H \subset X$ is called $\tilde{\pi}$*-asymptotically stable*, if it is both a weak $\tilde{\pi}$-attractor and $\tilde{\pi}$-orbitally stable.

Theorem 3.1. *If* $H \subset X$ *is* $\tilde{\pi}$*-asymptotically stable, then* H *is a* $\tilde{\pi}$*-attractor.*

Proof. By Lemma 3.1, it is enough to prove that $\widetilde{P}^+_W(H) \subset \widetilde{P}^+(H)$. Let $x \in \widetilde{P}^+_W(H)$ and U be an arbitrary neighborhood of H. By the $\tilde{\pi}$-stability of H, we can find a positively $\tilde{\pi}$-invariant neighborhood V of H such that $V \subset U$ and $V \subset \widetilde{P}^+_W(H)$. Let $\tau \in \mathbb{R}_+$ be such that $\tilde{\pi}(x, \tau) \in V$. Since V is positively $\tilde{\pi}$-invariant, we have $\tilde{\pi}(x, [\tau, +\infty)) \subset V \subset U$. Thus $x \in \widetilde{P}^+(H)$ and therefore $\widetilde{P}^+(H) = \widetilde{P}^+_W(H)$ is a neighborhood of H. □

The next corollary follows from Lemma 3.1 and Theorem 3.1.

Corollary 3.1. *A set* $\mathrm{H} \subset \mathrm{X}$ *is* $\tilde{\pi}$*-asymptotically stable if and only if it is both* $\tilde{\pi}$*-orbitally stable and a* $\tilde{\pi}$*-attractor.*

Theorem 3.2. *Suppose* X *is locally connected. Let* H *be a non-empty compact subset of* X *and suppose every component* A *of* H *has the property* $\mathrm{I}(\overline{A}) \subset A$, *and that* $\widetilde{\mathrm{P}}^+(\mathrm{H})$ *is open. Then* H *is* $\tilde{\pi}$*-asymptotically stable if and only if* H *has a finite number of components each of them* $\tilde{\pi}$*-asymptotically stable.*

The proof of Theorem 3.2 follows from the lemmas below in the same manner as the non-impulsive case. See, for instance, [10, p. 61].

Lemma 3.3. *Suppose* X *is locally connected,* $\mathrm{H} \subset \mathrm{X}$ *is a* $\tilde{\pi}$*-attractor,* $\widetilde{\mathrm{P}}^+(\mathrm{H})$ *is open and every component* A *of* $\widetilde{\mathrm{P}}^+(\mathrm{H})$ *has the property* $\mathrm{I}(\overline{A}) \subset A$. *If* A_1 *is an* I*-invariant component of* $\widetilde{\mathrm{P}}^+(\mathrm{H})$, *then* $\mathrm{H}_1 := A_1 \cap \mathrm{H}$ *is a non-empty* $\tilde{\pi}$*-attractor, with* $A_1 = \widetilde{\mathrm{P}}^+(\mathrm{H}_1)$.

Proof. Since X is locally connected, each component of $\widetilde{\mathrm{P}}^+(\mathrm{H})$ is open and the sets A_1 and $A_2 = \widetilde{\mathrm{P}}^+(\mathrm{H}) \setminus A_1$ are open and separated, that is, $\overline{A}_1 \cap A_2 = A_1 \cap \overline{A}_2 = \emptyset$. Let $\mathrm{H}_2 = A_2 \cap \mathrm{H}$. Notice that $\mathrm{H} = \mathrm{H}_1 \cup \mathrm{H}_2$. We will show that $\mathrm{H}_1 \neq \emptyset$.

If H_1 is empty, then A_2 is an open neighborhood of H. Let $x \in A_1$. We can suppose, without loss of generality, that $\phi(x) < +\infty$. Since $[0, \phi(x))$ is connected, it follows that $\tilde{\pi}(x, [0, \phi(x)) = \pi(x, [0, \phi(x))) \subset A_1$ and $x_1 = \pi(x, \phi(x)) \in A_1$. Because A_1 is open and I-invariant, we have $x_1^+ = \mathrm{I}(x_1) \in A_1$. In this manner, $\widetilde{\mathrm{K}}^+(x) \subset \overline{A}_1$. Therefore $\widetilde{\mathrm{K}}^+(x) \cap A_2 = \emptyset$ which is a contradiction, since A_2 is a neighborhood of H and $x \in \widetilde{\mathrm{P}}^+(\mathrm{H})$. Thus $\mathrm{H}_1 \neq \emptyset$. By the same argument $\mathrm{H}_2 \neq \emptyset$.

Since $\widetilde{\mathrm{P}}^+(\mathrm{H})$ is $\tilde{\pi}$-invariant, the same applies to A_1 and A_2. Besides, each point of A_1 is attracted by H_1. However, if any point $x \in A_1$ is attracted by H_2, then there exists $\tau \in \mathbb{R}_+$ such that $\tilde{\pi}(\tilde{\pi}(x, \tau), \mathbb{R}_+) \subset A_2$. But this contradicts the fact that A_1 is positively $\tilde{\pi}$-invariant. Therefore $A_1 \subset \widetilde{\mathrm{P}}^+(\mathrm{H}_1)$.

Since A_1 is open, $\tilde{\pi}$-invariant and encompasses H_1 and also $\widetilde{\mathrm{P}}^+(\mathrm{H}_1) \subset \widetilde{\mathrm{P}}^+(\mathrm{H}) = A_1 \cup A_2$, it follows that $A_1 \supset \widetilde{\mathrm{P}}^+(\mathrm{H}_1)$. As a consequence, $A_1 = \widetilde{\mathrm{P}}^+(\mathrm{H}_1)$ and H_1 is a $\tilde{\pi}$-attractor. □

Lemma 3.4. *Let* H_1 *and* H_2 *be separated by neighborhoods. If* $\mathrm{H}_1 \cup \mathrm{H}_2$ *is* $\tilde{\pi}$*-asymptotically stable, so are* H_1 *and* H_2. *However* $\widetilde{\mathrm{P}}^+(\mathrm{H}_1)$ *and* $\widetilde{\mathrm{P}}^+(\mathrm{H}_2)$ *are disjoint.*

Proof. It is enough to apply the proof of [10, Lemma 6.12] to the impulsive case with a few changes. □

We say that the orbit $\widetilde{\mathrm{C}}^+(x)$, $x \in \mathrm{X}$, *uniformly approximates* its limit set $\widetilde{\mathrm{L}}^+(x)$, whenever for every $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that $\widetilde{\mathrm{L}}^+(x) \subset B(\tilde{\pi}(x, [t, t+T]), \varepsilon)$ for all $t \in \mathbb{R}_+$.

The next theorem says that if the orbit of a point in X has compact closure and minimal limit set, then it is possible to uniformly approximate this orbit and its limit set. The converse is also true. For the non-impulsive case, see [1].

Theorem 3.3. *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system. Suppose* $\widetilde{\mathrm{K}}^+(p)$ *is compact,* $p \in \mathrm{X} \setminus \mathrm{M}$ *and* $\widetilde{\mathrm{L}}^+(p) \cap \mathrm{M} = \emptyset$. *Then* $\widetilde{\mathrm{L}}^+(p)$ *is minimal if and only if* $\widetilde{\mathrm{C}}^+(p)$ *uniformly approximates, its limit set* $\widetilde{\mathrm{L}}^+(p)$.

Proof. It is known (Kaul [6, p. 122]) that $\widetilde{\mathrm{L}}^+(p)$ is closed. Thus it is compact as it is a subset of the compact set $\widetilde{\mathrm{K}}^+(p)$. The set M is closed, so there is $\beta > 0$ such that $B(\widetilde{\mathrm{L}}^+(p), \beta) \cap \mathrm{M} = \emptyset$. Moreover, there is $s > 0$ such that $\tilde{\pi}(p,t) \notin \mathrm{M}$ for $t > s$. Thus, we have $\tilde{\pi}(\tilde{\pi}(p,s),t) = \pi(\tilde{\pi}(p,s),t)$, that is from some moment the trajectory goes without impulses.

Suppose $\widetilde{\mathrm{C}}^+(p)$ does not uniformly approximate its limit set $\widetilde{\mathrm{L}}^+(p)$. Then there are ε, $0 < \varepsilon < \beta$, a sequence of intervals $\{(t_n, \tau_n)\}$ and a sequence $\{y_n\}$ in $\widetilde{\mathrm{L}}^+(p)$ such that $t_n \to +\infty$, $(\tau_n - t_n) \to +\infty$, $y_n \to y \in \widetilde{\mathrm{L}}^+(p)$ and $y_n \notin B(\tilde{\pi}(p, [t_n, \tau_n]), \varepsilon)$.

We can assume, without loss of generality, that $\rho(y_n, y) < \varepsilon/3$, for every n. Thus, for arbitrary n, we have

$$\rho\big(y, \tilde{\pi}\big(p, [t_n, \tau_n]\big)\big) \geqslant \rho\big(y_n, \tilde{\pi}\big(p, [t_n, \tau_n]\big)\big) - \rho(y_n, y) > \varepsilon - \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Now we consider the sequence of points $\{\omega_n\}$, where $\omega_n = \tilde{\pi}(p, t'_n)$, with $t'_n = (t_n + \tau_n)/2$. It is clear that $t'_n \to +\infty$. We can assume, without loss of generality, that $\tilde{\pi}(p, t'_n) \to z \in \widetilde{\mathrm{L}}^+(p)$, since $\widetilde{\mathrm{K}}^+(p)$ is compact.

Because $\widetilde{\mathrm{L}}^+(p)$ is minimal and $z \notin \mathrm{M}$, we have $\widetilde{\mathrm{L}}^+(p) = \widetilde{\mathrm{K}}^+(z)$. Besides, $\{y_n\} \subset \widetilde{\mathrm{L}}^+(p) = \widetilde{\mathrm{K}}^+(z)$. Therefore, given n, there exists a sequence $\{\lambda^n_k\} \subset \mathbb{R}_+$ such that $\lambda^n_k \to +\infty$ as $k \to +\infty$, and

$$\tilde{\pi}\big(z, \lambda^n_k\big) \to y_n, \quad \text{as } k \to +\infty,$$

Hence $\rho(\tilde{\pi}(z, \lambda^n_k), y_n) < \varepsilon$, for sufficiently large k. And since $y_n \to y$, there exists a sequence $\{n_k\}$ of positive numbers such that

$$\rho\big(\tilde{\pi}\big(z, \lambda^k_{n_k}\big), y\big) < \frac{\varepsilon}{3}, \quad \text{for } n_k > k,$$

for k sufficiently large.

Let us choose n_M such that $\rho(\tilde{\pi}(z, \lambda^M_{n_M}), y) < \varepsilon/3$. Then the continuity of π implies there exists $\sigma > 0$ such that if $\rho(z, w) < \sigma$, then

$$\rho\big(\tilde{\pi}\big(z, \lambda^M_{n_M}\big), \tilde{\pi}\big(w, \lambda^M_{n_M}\big)\big) = \rho\big(\pi\big(z, \lambda^M_{n_M}\big), \pi\big(w, \lambda^M_{n_M}\big)\big) < \frac{\varepsilon}{3}.$$

Taking $\overline{N}$ sufficiently large such that $\rho(z, \omega_{\overline{N}}) < \sigma$ and $\lambda^M_{n_M} < (\tau_{\overline{N}} - t_{\overline{N}})/2$, we have

$$\rho\big(\tilde{\pi}\big(z, \lambda^M_{n_M}\big), \tilde{\pi}\big(\omega_{\overline{N}}, \lambda^M_{n_M}\big)\big) = \rho\big(\pi\big(z, \lambda^M_{n_M}\big), \pi\big(\omega_{\overline{N}}, \lambda^M_{n_M}\big)\big) < \frac{\varepsilon}{3}$$

and then

$$\rho\big(y, \tilde{\pi}\big(\omega_{\overline{N}}, \lambda^M_{n_M}\big)\big) \leqslant \rho\big(y, \tilde{\pi}\big(z, \lambda^M_{n_M}\big)\big) + \rho\big(\tilde{\pi}\big(z, \lambda^M_{n_M}\big), \tilde{\pi}\big(\omega_{\overline{N}}, \lambda^M_{n_M}\big)\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3},$$

But

$$\tilde{\pi}\big(\omega_{\overline{N}}, \lambda^M_{n_M}\big) = \tilde{\pi}\big(\tilde{\pi}\big(p, t'_{\overline{N}}\big), \lambda^M_{n_M}\big) = \tilde{\pi}\big(p, t'_{\overline{N}} + \lambda^M_{n_M}\big),$$

with

$$t_{\overline{N}} < \frac{t_{\overline{N}} + \tau_{\overline{N}}}{2} < t'_{\overline{N}} + \lambda^M_{n_M} < \frac{t_{\overline{N}} + \tau_{\overline{N}}}{2} + \frac{\tau_{\overline{N}} - t_{\overline{N}}}{2} = \tau_{\overline{N}}.$$

Thus $\tilde{\pi}(\omega_{\overline{N}}, \lambda^M_{n_M}) \in \tilde{\pi}(p, [t_n, \tau_n])$ which is a contradiction, since

$$\rho\big(y, \tilde{\pi}\big(p, [t_n, \tau_n]\big)\big) > \frac{2\varepsilon}{3}.$$

Therefore $\widetilde{\mathrm{C}}^+(p)$ uniformly approximates its limit set $\widetilde{\mathrm{L}}^+(p)$.

Conversely, suppose $\widetilde{\mathrm{L}}^+(p)$ is not minimal. Then $\widetilde{\mathrm{L}}^+(p) \neq \widetilde{\mathrm{K}}^+(y)$, for some $y \in \widetilde{\mathrm{L}}^+(p) \setminus \mathrm{M}$. Then there exists $z \in \widetilde{\mathrm{L}}^+(p)$ such that $z \notin \widetilde{\mathrm{K}}^+(y)$.

Let $\varepsilon = \rho(z, \widetilde{\mathrm{K}}^+(y)) > 0$. By the uniform approximation, there exists $T > 0$ such that

$$\widetilde{\mathrm{L}}^+(p) \subset B\left(\tilde{\pi}\big(p, [t, t+T]\big), \frac{\varepsilon}{2}\right), \quad \text{for } t \geqslant 0. \tag{1}$$

Besides, there is δ, $0 < \delta < \beta$, such that if $\rho(y, w) < \delta$, then

$$\rho\big(\tilde{\pi}(y, t), \tilde{\pi}(w, t)\big) = \rho\big(\pi(y, t), \pi(w, t)\big) < \frac{\varepsilon}{2}, \quad \text{for } t < T. \tag{2}$$

Since $y \in \widetilde{\mathrm{L}}^+(p)$, there is a sequence $\{t_n\} \subset \mathbb{R}_+$ such that $\tilde{\pi}(p, t_n) \to y$. Therefore there is $N > 0$ sufficiently large such that

$$\rho\big(\tilde{\pi}(p, t_N), y\big) < \delta. \tag{3}$$

Since (1) holds we have, in particular, that $\widetilde{\mathrm{L}}^+(p) \subset B(\tilde{\pi}(\tilde{\pi}(p, t_N), [0, T]), \varepsilon/2)$. And because $z \in \widetilde{\mathrm{L}}^+(p)$, it follows that

$$\rho(z, x) < \frac{\varepsilon}{2}, \quad \text{for } x = \tilde{\pi}\big(\tilde{\pi}(p, t_N), \tau\big), \quad \text{for some } \tau \in [0, T].$$

It follows from (3), in view of (2), that

$$\rho\big(\tilde{\pi}\big(\tilde{\pi}(p, t_N), \tau\big), \tilde{\pi}(y, \tau)\big) < \frac{\varepsilon}{2}.$$

Hence

$$\rho\big(z, \tilde{\pi}(y, \tau)\big) \leqslant \rho(z, x) + \rho\big(x, \tilde{\pi}(y, \tau)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which contradicts the fact that $\rho(z, \widetilde{\mathrm{K}}^+(y)) = \varepsilon$. Therefore $\widetilde{\mathrm{L}}^+(p)$ is minimal and the proof is complete. □

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