KAZHDAN'S PROPERTY (T)

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1.1. History and motivations.

Property (T) was introduced by D. Kazhdan in 1967, in a 3-page paper. He was interested in locally symmetric Riemannian spaces of the form $\Gamma\backslash X$, where X is a Riemannian symmetric space of the non-compact type (e.g. $X = SL_n(\mathbb{R})/SO(n)$), and Γ is a discrete subgroup of $\mathrm{Isom}(X)$, acting freely on X, and such that $\Gamma\backslash X$ has finite volume. He was interested in properties like:

- Is $\pi_1(\Gamma\backslash X) = \Gamma$ finitely generated? ($\Leftrightarrow$ does Γ admit in X a fundamental domain which is a finite polyhedron?) [Yes if $\Gamma\backslash X$ compact, not obvious otherwise]
- Is the first Betti number $b_1(X)$ zero? ($b_1(X) = \dim \mathrm{Hom}(\Gamma, \mathbb{R})$)

Kazhdan proved that, if $\mathrm{rk}\, X \geq 2$, the answer is yes for both questions. Observe that Γ is a lattice in $G = \mathrm{Isom}\, X$, a semi-simple Lie group. Kazhdan singled out a representation-theoretic property of locally compact groups and proved:

- If G has property (T), then G is compactly generated and $\mathrm{Hom}(G, \mathbb{R}) = \{0\}$.
- If Γ is a lattice in G and G has property (T), so does Γ.
- If G is a simple Lie group with $\mathrm{rk}_{\mathbb{R}} G \geq 2$ (for $G = SL_n(\mathbb{R})$ this means $n \geq 3$), then G has (T).

1.2. Basic definitions and results

We will consider unitary representations π of locally compact groups G on Hilbert spaces $\mathcal{H}$. So, if $U(\mathcal{H})$ is the unitary group of $\mathcal{H}$, then $\pi : G \to U(\mathcal{H})$ is a group homomorphism, <u>strongly continuous</u> (i.e. $G \times \mathcal{H} \to \mathcal{H} : (g, \xi) \mapsto \pi(g)\xi$ is continuous).

We say that π has <u>fixed vectors</u> if $\exists\, \xi \in \mathcal{H}, \xi \neq 0$ such that $\pi(g)\xi = \xi$ $\forall g \in G$.

<u>Example</u>: The left regular representation of G on $L^2(G)$ is defined (Haar measure d

by $(\lambda_G(g)\xi)(x) = \xi(g^{-1}x)$

A function ξ is $\lambda_G(G)$-invariant $\iff$ ξ is constant; but $L^2(G)$ contains non-zero constant $\iff$ G is compact.

Def: π *almost has invariant vectors* if $\forall \varepsilon > 0$, $\forall K \subset G$ compact subset, $\exists \xi \in \mathcal{H}$ s.t. $\max_{g \in K} \|\pi(g)\xi - \xi\| < \varepsilon \|\xi\|$.

If G is σ-compact, this is equivalent to: $\exists (\xi_n)_{n \geq 1}$ sequence of unit vectors in $\mathcal{H}$ s.t. $(g \mapsto \|\pi(g)\xi_n - \xi_n\|)_{n \geq 1}$ converges to 0 unif. on compact subsets of G.

⚠ If $\dim \pi < +\infty$, then π almost has invariant vectors $\iff$ π has invariant vectors. Indeed, the unit sphere of $\mathcal{H}$ is then compact, so from $(\xi_n)_{n \geq 1}$ extract a convergent subsequence $\xi_{n_k} \to \xi$. Then ξ is a fixed vector.

Example: $G = \mathbb{R}$; $\lambda_{\mathbb{R}}$ almost has invariant vectors.
Set $\xi_n = \frac{\mathbb{1}_{[0,n]}}{\sqrt{n}}$ so that $\|\xi_n\| = 1$. For $x \in [-a, a]$:

$$\|\lambda(x)\xi_n - \xi_n\|_2^2 \leq \frac{2a}{n} \xrightarrow[n \to \infty]{} 0,$$

so $\lambda_{\mathbb{R}}$ almost has invariant vectors, but no invariant vectors.

Prop (Reiter 1964): λ_G almost has invariant vectors $\iff$ G is amenable. ▨

(a group G is *amenable* if every affine action of G on a compact convex set has a fixed point: this defines an important class of groups containing compact groups, abelian groups, solvable groups...).

A criterion for invariant vectors

Prop: Suppose that $\exists \xi \in \mathcal{H}$, $\|\xi\| = 1$, such that $\sup_{g \in G} \|\pi(g)\xi - \xi\| < \sqrt{2}$. Then π has invariant vectors.

Pf: Set $\varepsilon > 0$ such that $\|\pi(g)\xi - \xi\| \leq \sqrt{2} - \varepsilon$ $\forall g \in G$. Then

$$2 - 2\,\mathrm{Re}\langle \pi(g)\xi | \xi \rangle \leq (\sqrt{2} - \varepsilon)^2$$

$\Rightarrow \alpha = 2 - (\sqrt{2} - \varepsilon)^2 \leq \mathrm{Re}\langle \pi(g)\xi | \xi \rangle$. So for η in the closed

convex span $\overline{\mathrm{conv}\,\pi(G)\xi}$: $\alpha \leq \mathrm{Re}\langle\eta|\xi\rangle$.
In particular $0 \notin \overline{\mathrm{conv}\,\pi(G)\xi}$. The projection of 0 on $\overline{\mathrm{conv}\,\pi(G)\xi}$ is then a $\pi(G)$-invariant non-zero vector.

[Def: G has property (T) if every representation π almost having invariant vectors, has non-zero invariant vectors.

Example: Compact groups have (T): if G is compact, and π almost has invariant vectors, then π has $(\sqrt{2}-\varepsilon, G)$-invariant vectors, so it has fixed vectors.

Prop: Prop (T) + amenable $\Leftrightarrow$ compact.

Proof: λ_G almost has invariant vectors, so has invariant vectors, so G is compact.

[Prop: If $\alpha: G \to H$ is a continuous homom, G prop (T), then $\overline{\alpha(G)}$ has (T)

Pf: If π is a rep of $\overline{\alpha(G)}$ with almost invariant vectors, then so is $\pi\circ\alpha$ as a rep. of G; by property (T), $\pi\circ\alpha$ has some invariant vector ξ. By density of $\alpha(G)$ in $\overline{\alpha(G)}$, ξ is $\overline{\alpha(G)}$-fixed ☑

[Cor: If $\alpha: G \to H$ is continuous, G has (T), H is amenable, then $\overline{\alpha(G)}$ is compact

Pf: $\overline{\alpha(G)}$ is (T) + amenable.

[Cor: If G has (T), then $G/\overline{[G,G]}$ is compact. In particular $\mathrm{Hom}(G, \mathbb{R}) = \{0\}$

Pf: $G/\overline{[G,G]}$ is abelian + (T), hence compact. Every hom. $G \to \mathbb{R}$ factors through $G/\overline{[G,G]}$, hence is constant. ☑

Example: The free group $\mathbb{F}_n$ does not have (T), since $\mathbb{F}_n/[\mathbb{F}_n,\mathbb{F}_n] \simeq \mathbb{Z}^n$.

Prop: Assume H has finite co-volume in G (i.e. there is a G-invariant prob. measure on G/H). If G has (T), so does H.

Rem: Converse true, and due to H.C. Wang (1971).

Pf: Let π be a rep. of H, with almost invariant vectors. Let $\sigma = \mathrm{Ind}_H^G \pi$ be the induced rep. Recall:

$\xi \in \mathcal{H}_\pi$ D dom fond pour G/H

$f_\xi(g) = \pi(h)^{-1}\xi$ si $g \in Dh$.

$\|f_\xi\| = \|\xi\|$

$$\|\sigma(k) f_\xi - f_\xi\|_2^2 = \int_{G/H} \|f_\xi(k^{-1}x) - f_\xi(x)\|^2 dx$$

$$= \int_{G/H} \|$$

$f_\xi(k^{-1}g) = \pi(h)^{-1}\xi$ si $g \in kDh$.

σ is a rep of $\{f : G \to \mathcal{H}_\pi : f(gh) = \pi(h)^{-1} f(g)\ \forall g \in G, h \in H$, $\int_{G/H} \|f(x)\|^2 dx < +\infty\}$.

G acts by
$\sigma(g) f(y) = f(g^{-1}y)$.

~~Assume we know that~~ But σ almost has invariant vectors (Fell 1960). Then σ has a non-zero fixed vector, i.e. a non-zero constant map $f : G \to \mathcal{H}_\pi$.

Then $f(h) = f(1) = \pi(h)^{-1} f(1)$, i.e. $f(1)$ is $\pi(H)$-fixed. ☒

Example: $SL_2(\mathbb{R})$ does not have (T), since $\mathbb{F}_2$ is a lattice without (T).

Prop: If G has (T), then G is compactly generated.

Proof. For every compact neighbourhood K of e in G, let U_K be the sgp. generated by K. Want to show: $U_K = G$ for some K. Note that U_K is open. Set π_K = rep. of G on $\ell^2(G/U_K)$, and $\pi = \bigoplus_K \pi_K$; π almost has invariant vectors; indeed, if C is compact in G, then $C \subset U_K$ for some K, and the Dirac mass δ_{U_K} (at identity coset of U_K) is $\pi(U_K)$-invariant. By property (T), π has fixed vectors. Hence some π_K has fixed vectors, i.e. $\exists$ non-zero constant function in $\ell^2(G/U_K)$. So $|G/U_K| < +\infty$, i.e. U_K has finite index in G. Enlarge K by representatives of the cosets of U_K in G, to get a compact generating set of G. ☒

1.3 Main examples:

Thm (Kazhdan) Let G be a simple Lie group of $\mathbb{R}$-rank ≥ 2. Then G has (T). (e.g. $SL_n(\mathbb{R})$, $n \geq 3$; $Sp_{2n}(\mathbb{R})$, $n \geq 2$).

Strategy of proof:
- Prove separately the result for $SL_3(\mathbb{R})$ and $Sp_4(\mathbb{R})$.
- Use that every group G of $\mathbb{R}$-rk ≥ 2 contains either $SL_3(\mathbb{R})$ or $Sp_4(\mathbb{R})$ (up to local isomorphism) as a closed subgroup H
- Find $X \in \text{Lie } \mathfrak{h}$, $X \neq 0$, s.t. $ad_{\mathfrak{g}}(X)$ is diagonalizable over $\mathbb{R}$.

Let $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda$ be decomposition as eigenspaces of X.
Set $\mathfrak{g}^+ = \bigoplus_{\lambda > 0} \mathfrak{g}_\lambda$ and $\mathfrak{g}^- = \bigoplus_{\lambda < 0} \mathfrak{g}_\lambda$; let $\mathfrak{n}$ be the Lie subalgebra generated by $\mathfrak{g}^+ \cup \mathfrak{g}^-$. We claim that $\mathfrak{n} = \mathfrak{g}$. First, since $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$,

we have that $\mathfrak{n}$ is an ideal. By simplicity, $\mathfrak{n} = \{0\}$ or $\mathfrak{n} = \mathfrak{g}$. The first case implies $\mathfrak{g} = \mathfrak{g}_0$, i.e. $\mathfrak{g} = \operatorname{Ker} \operatorname{ad}(X)$, i.e. X is in the centre of $\mathfrak{g}$; so $X = 0$, contrad. Set $a = \operatorname{Exp} X$.

For $Y \in \mathfrak{g}^\lambda$:

$a \operatorname{Exp} Y a^{-1} = \operatorname{Exp}(\operatorname{Ad}(a) Y) = \operatorname{Exp}(e^\lambda Y)$.

So for $n \in \mathbb{Z}$: $a^n (\operatorname{Exp} Y) a^{-n} = \operatorname{Exp}(e^{n\lambda} Y)$:

- if $\lambda > 0$, then $a^{-n}(\operatorname{Exp} Y) a^{n} \underset{n\to\infty}{\longrightarrow} e$
- if $\lambda < 0$, then $a^{n}(\operatorname{Exp} Y) a^{-n} \underset{n\to\infty}{\longrightarrow} e$.

If π is a rep of G, with almost invariant vectors, then $\pi|_H$ has a fixed vector ξ. In particular it is $\pi(a^n)$-fixed. By Mautner's lemma (see below), ξ is also fixed under $\pi(\operatorname{Exp} Y)$, $Y \in \mathfrak{g}_\lambda$. Since these generate all of G, ξ is $\pi(G)$-fixed ▨

Mautner's lemma (1951): Let G be locally compact group, π a representation of G. Let $x, (y_n)_{n\in\mathbb{N}}$ be elements of G such that $y_n x y_n^{-1} \underset{n\to\infty}{\longrightarrow} e$. If $\xi \in \mathcal{H}_\pi$ is such that $\pi(y_n)\xi = \xi$ $\forall n \geq 1$, then $\pi(x)\xi = \xi$.

Pf: $\langle \pi(x)\xi \mid \xi \rangle = \langle \pi(x)\pi(y_n^{-1})\xi \mid \pi(y_n^{-1})\xi \rangle$

$= \langle \pi(y_n x y_n^{-1})\xi \mid \xi \rangle \underset{n\to\infty}{\longrightarrow} \langle \pi(e)\xi \mid \xi \rangle = \langle \xi \mid \xi \rangle$

$\Rightarrow \langle \pi(x)\xi \mid \xi \rangle = \|\xi\|^2$. By the equality case of Cauchy-Schwarz: $\pi(x)\xi = \xi$ ▨

Sketch of proof of property (T) for $SL_3(\mathbb{R})$.

Def: Let G be a loc. compact group, H a closed subgroup. The pair (G, H) has relative property (T) if every rep of G, almost having invariant vectors, has H-invariant vectors.

Examples: 1) If H is compact, then (G, H) has relative (T).

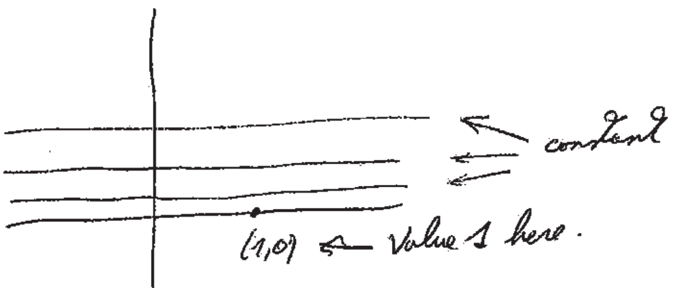
2) G has property (T) $\Leftrightarrow$ (G, G) has rel. (T).

Proposition: The pair $(\mathbb{R}^2 \rtimes SL_2(\mathbb{R}), \mathbb{R}^2)$ has relative (T).

Let us prove Prop (T) for $SL_3(\mathbb{R})$ modulo that prop.

Lemma: Let $N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{R} \right\} \subset SL_2(\mathbb{R})$. If $\pi \in Rep(SL_2(\mathbb{R}))$ has an N-invariant vector ξ, then ξ is $SL_2(\mathbb{R})$-fixed.

<u>Pf:</u> Set $A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}$, $N_- = \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} : n \in \mathbb{R} \right\}$.
It is enough to show that ξ is A-fixed. (Then Mautner implies that ξ is N_--fixed, and N, A, N_- generate $SL_2(\mathbb{R})$).
The function $g \mapsto \langle \pi(g)\xi | \xi \rangle$ is right N-invariant, so it descends to a function on $G/N \simeq \mathbb{R}^2 \setminus \{(0,0)\}$. As $\langle \pi(ng)\xi | \xi \rangle = \langle \pi(g)\xi | \pi(n^{-1})\xi \rangle = \langle \pi(g)\xi | \xi \rangle$, it is also left N-invariant, hence constant on horizontal lines on $\mathbb{R}^2 \setminus \{(0,0)\}$, except maybe the x-axis. By continuity, this function is also constant on the x-axis i.e. the orbit of $(1,0)$ under A. Hence $\langle \pi(a_t)\xi | \xi \rangle = 1$, hence $\pi(a_t)\xi = \xi$. ☐



Prop: $SL_3(\mathbb{R})$ has (T)

<u>Pf:</u> Let π be a rep. of $SL_3(\mathbb{R})$, almost having fixed vector.
Embed $\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ as: $\left\{ \begin{pmatrix} A & \begin{matrix} x \\ y \end{matrix} \\ 0 & 1 \end{pmatrix}, A \in SL_2(\mathbb{R}), (x,y) \in \mathbb{R}^2 \right\}$.
By rel (T), π has V-invariant vectors ξ, where $V = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : (x,y) \in \mathbb{R}^2$.
Embed $SL_2(\mathbb{R})$ into $SL_3(\mathbb{R})$ as $\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix}$, and as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$
By previous lemma, ξ is invariant by these two copies of $SL_2(\mathbb{R})$, which together generate $SL_3(\mathbb{R})$. ☐

<u>Proof of rel (T) for $(\mathbb{R}^2 \rtimes SL_2(\mathbb{R}), \mathbb{R}^2)$:</u>
Let π be a rep of $\mathbb{R}^2 \rtimes SL_2(\mathbb{R})$; then by the spectral thm there exists a projection valued measure E on $\widehat{\mathbb{R}^2}$ such that

$$\pi|_{\mathbb{R}^2} = \int_{\widehat{\mathbb{R}^2}} \lambda(.)\, dE(\lambda),$$

i.e. $\pi(x,y) = \int_{\widehat{\mathbb{R}^2}} e^{2\pi i(sx+ty)}\, dE(s,t)$; moreover for $g \in SL_2(\mathbb{R})$, $B \subset \widehat{\mathbb{R}^2}$
$E(gB) = \pi(g) E(B) \pi(g^{-1})$. For ξ a unit vector in $\mathcal{H}$, then $\mu_\xi(B) = \langle E(B)\xi | \xi \rangle$

is a prob. measure on $\mathbb{R}^2$, whose Fourier transform is $\langle \pi(x,y)\xi|\xi\rangle$. Observe: $\pi|_{\mathbb{R}^2}$ has no invariant vector $\Leftrightarrow E\{(0,0)\}=0$. Also, for $g\in SL_2(\mathbb{R})$:

$$g_*\mu_\xi(B)=\mu_\xi(g^{-1}B)=\langle E(g^{-1}B)\xi|\xi\rangle=\langle E(B)\pi(g)\xi|\pi(g)\xi\rangle$$

$\Rightarrow g_*\mu_\xi=\mu_{\pi(g)\xi}$.

Hence

$$|\mu_{\pi(g)\xi}(B)-\mu_\xi(B)|=|\langle E(B)\pi(g)\xi|\pi(g)\xi\rangle-\langle E(B)\xi|\pi(g)\xi\rangle$$
$$+\langle E(B)\xi|\pi(g)\xi\rangle-\langle E(B)\xi|\xi\rangle|\leq 2\|\pi(g)\xi-\xi\|.$$

So assume by contradiction that π has almost invariant vectors, but no $\pi|_{\mathbb{R}^2}$-invariant vectors. Let $(\xi_n)_{n\geq 1}$ be a sequence of almost invariant vectors. Then $|\mu_{\xi_n}-\mu_{\pi(g)\xi_n}|\underset{n\to\infty}{\to}0$ (in total variation) and $\mu_{\xi_n}\{(0,0)\}=0$. View μ_{ξ_n} as prob. measure on $\mathbb{R}^2\setminus\{(0,0)\}$, and push it forward to a prob. measure ν_n on the projective line $P^1(\mathbb{R}^2)$. Then $(\nu_n)_{n\geq 1}$ is an almost $SL_2(\mathbb{R})$-invariant of prob. measures on P^1; by compactness we may find a limit point ν, i.e. an $SL_2(\mathbb{R})$-invariant prob. measure. Contradiction!

2. Affine isometric actions

2.1. 1-cohomology.

Let $\mathcal{H}$ be a real Hilbert space. Assume we have an orthogonal rep. $\pi: G\to O(\mathcal{H})$. We want to try to deform it inside the bigger group $\mathrm{Isom}\,\mathcal{H}$. By Mazur-Ulam theorem, every isometry of $\mathcal{H}$ is affine (easy in Hilbert space: metric characterization of segments.

$$[a,b]=\{c\in\mathcal{H}: \|a-c\|+\|c-b\|=\|a-b\|\}).$$

Then $\mathrm{Isom}\,\mathcal{H}=\mathcal{H}\rtimes O(\mathcal{H})$.

We are interested in deformations of π, i.e. homomorphisms

$$\alpha: G\to \mathrm{Isom}\,\mathcal{H}: g\mapsto \alpha(g)$$

where $\alpha(g)v=\underbrace{\pi(g)v}_{\text{linear part}}+\underbrace{b(g)}_{\text{translation part}}$.

Multiplicativity of α translates into 1-cocycle relation for b: $\quad b(gh)=\pi(g)b(h)+b(g)$.

Dictionary:

Actions	1-Cocycles
α is an action with linear part π	$b(gh) = \pi(g)b(h) + b(g)$
Deformation space	$Z^1(G,\pi)$, vector space of 1-cocycles
Trivial deformations (conjugate π by a translation t_{-v})	$B^1(G,\pi)$ vector space of 1-coboundaries $B^1(G,\pi) = \{b \in Z^1(G,\pi) : b(g) = \pi(g)v - v$ for $v \in$
~~Deformat~~ Actions modulo trivial ones	$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi)$
Actions with a bounded orbit	Bounded cocycles.

Example: If π is the trivial rep on $\mathcal{H}$, then
$$Z^1(G,\pi) = H^1(G,\pi) = \mathrm{Hom}(G,\mathcal{H}).$$

Def: G has property (FH) if every affine isometric action of G on a Hilbert space, has a fixed point. ($\Leftrightarrow H^1(G,\pi) = 0 \ \forall \pi \in \mathrm{Rep}\, G$).

Thm (Guichardet 1972): If G is σ-compact, then (FH) $\Rightarrow$ (T)

Proof: By contraposition, assume that G does not have (T), so there exists a rep. π almost having invariant vectors, but no invariant vectors. Endow $Z^1(G,\pi)$ with the topology of unif. convergence on compact subsets. As G is σ-compact, $Z^1(G,\pi)$ is a Fréchet space. Consider the map
$\partial : \mathcal{H} \to Z^1(G,\pi) : v \mapsto (g \mapsto \pi(g)v - v)$. Then

- $\mathrm{Im}\, \partial = B^1(G,\pi)$
- ∂ is injective (no invariant vector)
- $\|\partial v(g)\| \le 2\|v\| \Rightarrow \partial$ is continuous.

Claim: $B^1(G,\pi)$ is not closed in $Z^1(G,\pi)$ (and so $H^1(G,\pi) \neq 0$).

Assume $B^1(G,\pi) = \mathrm{Im}\,\partial$ is closed. Since $B^1(G,\pi)$ is Fréchet, the closed graph thm says that $\partial^{-1} : B^1(G,\pi) \to \mathcal{H}$ is continuous, i.e. ~~$\forall v \in \mathcal{H}$~~: $\exists C > 0$, $K \subset G$ compact $\forall v \in \mathcal{H}$: $\|v\| \le C \sup_{g \in K} \|\pi(g)v - v\|$

So for $\|v\| = 1$: $\sup_{g \in K} \|\pi(g)v - v\| \geq \frac{1}{C}$ $\forall v \in S^1(\mathcal{H})$, so π does not have almost invariant vectors.

~~2.2.1~~ Bounded cocycles.

It is clear that every coboundary is bounded ($\Leftrightarrow$ every action with a fixed point has bounded orbits). We show that the converse holds

Lemma: Every bounded cocycle is a coboundary ($\Leftrightarrow$ every action with bounded orbits, has a fixed point).

Pf: We use the fact that every non-empty bounded subset C of a Hilbert space, has a unique smallest closed ball containing it. If C is invariant under a group of isometries, by uniqueness the circumcenter is fixed.

We apply this to a bounded orbit $\alpha(G)v$ ~~bounded~~ $\square$

2.2. Schönberg's theorem and consequences:

Def: A kernel Φ on a set X ~~is~~ (i.e. $\Phi: X \times X \to \mathbb{C}$), is *positive-definite* if $\forall x_1, \dots, x_n \in X, \forall c_1, \dots, c_n \in \mathbb{C}$:

$$\sum_{i,j} c_i \overline{c_j}\, \Phi(x_i, x_j) \geq 0.$$

Example: $f: X \to \mathcal{H}$, $\Phi(x,y) = \langle f(x) | f(y) \rangle$.
By GNS construction, every positive-definite kernel is of this form.

Thm (Schönberg 1934): If $b \in Z^1(G,\pi)$, then $\forall t > 0$: the kernel $\Phi_t(g,h) = e^{-t\|b(g)-b(h)\|^2}$ is positive-definite on G, which is G-invariant: $\Phi_t(sg, sh) = \Phi_t(g,h)$.

Pf: Set $\Psi(g,h) = \|b(g)\|^2 + \|b(h)\|^2 - \|b(g)-b(h)\|^2 = 2\langle b(g)|b(h)\rangle$ a positive-definite kernel. So

$$\exp(-\|b(g)-b(h)\|^2) = \underbrace{\exp \Psi(g,h)}_{\text{pos.-def.}}\ \underbrace{\exp(-\|b(g)\|^2)\exp(-\|b(h)\|^2)}_{\text{pos.-def.}}$$

$\exp \Psi(g,h) = \sum_{n=0}^{\infty} \frac{\Psi(g,h)^n}{n!}$ and the set of pos. def. kernels is a closed convex cone

$$\sum_{i,j} c_i \overline{c_j} \exp(-\|b(g_i)\|^2)\exp(-\|b(g_j)\|^2) = \left|\sum_i c_i \exp(-\|b(g_i)\|^2)\right|^2 \geq 0$$

G-invariance: $\|b(sg) - b(sh)\|^2 = \|\pi(s)b(g) + b(s) - \pi(s)b(h) - b(s)\|^2$
$= \|\pi(s)(b(g)-b(h))\|^2 = \|b(g)-b(h)\|^2$. $\square$

As a consequence of the GNS-construction, we then have

Prop: For $b \in Z^1(G,\pi)$, $\alpha(g)v = \pi(g)v + b(g)$, $t > 0$: there exists a complex Hilbert space $\mathcal{H}_t$, a continuous mapping $F_t: \mathcal{H} \to S^1(\mathcal{H}_t)$, and a unitary rep. π_t of G on $\mathcal{H}_t$, such that:

i) $\langle F_t(\xi) | F_t(\eta) \rangle = e^{-t\|\xi-\eta\|^2}$ $\forall \xi, \eta \in \mathcal{H}$

ii) $\pi_t(g) F_t(\xi) = F_t(\alpha(g)\xi)$ $\forall \xi \in \mathcal{H}, g \in G$.

iii) The linear span of $F_t(\mathcal{H})$ is dense in $\mathcal{H}_t$ ☒

Rem: Symmetric Fock space allows for an explicit construction of $\mathcal{H}_t$.

Observe: α has a fixed point $\Leftrightarrow \pi_t$ has a non-zero fixed vector.

If $\alpha(g)\xi = \xi$, then $\pi_t(g) F_t(\xi) = F_t(\xi)$. For the converse, assume α has no fixed point, i.e. b is unbounded. We show: if $\pi_t(g)\eta = \eta$ for $\eta \in \mathcal{H}_t$, then $\eta = 0$. Then for $\xi \in \mathcal{H}$:

$$\langle F_t(\xi) | \eta \rangle = \langle F_t(\xi) | \pi_t(g)\eta \rangle = \langle \pi_t(g^{-1}) F_t(\xi) | \eta \rangle$$
$$= \langle F_t(\alpha(g)\xi) | \eta \rangle. \quad \forall g \in G.$$

Take $g_n \in G$ such that $\|b(g_n)\| \xrightarrow[n\to\infty]{} \infty$. Then $F_t(\alpha(g_n)\xi) \xrightarrow[n\to\infty]{\text{weakly}} 0$, and so $\langle F_t(\xi) | \eta \rangle = 0$. Since $F_t(\mathcal{H})$ is total in $\mathcal{H}_t$, this implies $\eta = 0$.

Indeed: $\langle F_t(\alpha(g_n)\xi) | F_t(\xi) \rangle = e^{-t\|\alpha(g)\xi_n - \xi\|^2} \xrightarrow[n\to\infty]{} 0$.

Thm (DELORME 1973): If G has property (T), then G has (FH)

Proof: Suppose that G does not have (FH). So find $b \in Z^1(G, \pi)$, unbounded. Form the above family of representations π_t, and form $\pi = \bigoplus_{n=1}^{\infty} \pi_{1/n}$. Then by the above remark, π has no fixed vector. But π almost has invariant vectors. For $\varepsilon > 0$, $K \subset G$; $g \in K$:

$$\|\pi_t(g) F_t(0) - F_t(0)\|^2 = \|F_t(b(g)) - F_t(0)\|^2 =$$
$$2 - 2e^{-t\|b(g)\|^2} \leq 2t\|b(g)\|^2 \leq 2t \max_{g\in K} \|b(g)\|^2 < \varepsilon \text{ for } t$$

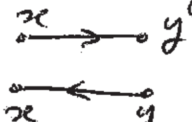
small enough. ☒

2.3. Applications.

Framework: Let (X, d) be a metric space, assume there exists $b \in Z^1(\mathrm{Isom}(X), \pi)$ such that $\|b(g)\| = F(d(gx_0, x_0))$, for some $x_0 \in X$ and $F: \mathbb{R}^+ \to \mathbb{R}$ a function such that $\lim_{t\to+\infty} F(t) = +\infty$. Assume that a proper

(T) group G acts by isometries on X. Then G has bounded orbits on X (reason: $g \mapsto \|b(g)\|$ is bounded on G, hence $g \mapsto d(gx_0, x_0)$ is bounded on G). In good cases, one can go down from bounded orbits to fixed points on X.

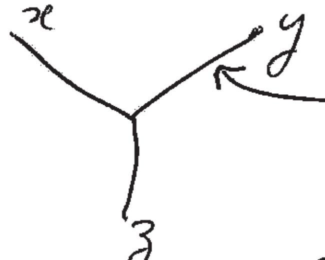
2.3.a) Trees.

Let X be a tree, i.e. a connected graph without circuit. V = set of vertices, $\mathbb{E}$ set of oriented edges (every geometric edge comes with two orientations 

). The group $\mathrm{Aut}(X)$ has a natural representation π on $\ell^2(\mathbb{E})$. Define $c: V \times V \to \ell^2(\mathbb{E})$: $(x,y) \mapsto c(x,y)$ where $c(x,y)(e) =$
- 0 if $e \notin [x,y]$ (unique geodesic from x to y)
- 1 if 

- -1 if 

Then:
- $\|c(x,y)\|_2^2 = 2\, d(x,y)$
- $\pi(g)\, c(x,y) = c(gx, gy)$
- $c(x,y) + c(y,z) = c(x,z)$:



Fix then a base vertex $x_0 \in V$, and set $b(g) = c(gx_0, x_0)$

Then $b(gh) = c(ghx_0, x_0) = c(ghx_0, gx_0) + c(gx_0, x_0)$
$= \pi(g)\, b(h) + b(g)$

So $b \in Z^1(\mathrm{Aut}\,X, \pi)$ and $\|b(g)\| = \sqrt{2\, d(gx_0, x_0)}$

Lemma (Serre): If a group G acts on X with a bounded orbit, then G fixes a vertex or an edge.

Pf: Let T be the smallest subtree containing a bounded orbit (so T is G-invariant). Say $\mathrm{diam}\, T = n$. Remove from T all terminal vertices. Get a G-invariant subtree T' of diameter $n-2$. Iterate. At the end, get an invariant subtree of diameter 1 (= an edge) or 0 (= a vertex) ▨

Def (Serre) A group G has property (FA) if every action on a tree fixes an edge or a vertex. We just proved:

<u>Proposition</u>: (Watatani 1981): (T) $\Longrightarrow$ (FA)

<u>Cor</u>: Assume X is locally finite (so that $\mathrm{Aut}\,X$ is locally compact, for topology of pointwise convergence). If G has (T), any continuous homom $G \to \mathrm{Aut}(X)$ has relatively compact image ☒

2.3. b) Real hyperbolic spaces.

Let $X_n = \mathbb{H}^n(\mathbb{R})$ be real hyperbolic space of dimension n. Then $\mathrm{Isom}(X_n) \simeq O(n,1)$ (group of isometries of quadratic form $x_1^2 + \dots + x_n^2 - x_{n+1}^2$ on $\mathbb{R}^{n+1}$). A <u>hyperplane</u> in X_n is a totally geodesic subspace isometric to X_{n-1}, a <u>half-space</u> in X_n is one of the 2 connected components of the complement of a hyperplane. Set $\mathcal{H}_n$ = set of hyperplanes.

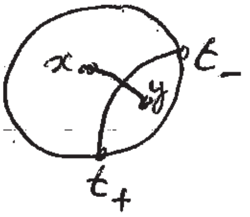
2 half-spaces

$\mathrm{Isom}\,X_n$ acts transitively on $\mathcal{H}_n$, with an invariant measure μ which is finite on compact subsets of $\mathcal{H}_n$.

For $x \in X_n$, let H_x be the set of half-spaces through x.

<u>Claim</u>: for $x, y \in X_n$: $H_x \,\Delta\, H_y$ (= the set of half-spaces separating x from y) is relatively compact in $\mathcal{H}_n$.

<u>Pf</u> for $n = 2$. Enough to see that the set of lines in X_2 separating x from y, is relatively compact. But a line in X_2 is determined by its endpoints, so the space of lines is $(S^1 \times S^1) \setminus \Delta \,/\, S_2$. And if (t_+, t_-) is close to Δ, the line (t_+, t_-) will not cross $[xy]$.



Now: define $c : X_n \times X_n \to L^2(\mathcal{H}_n, \mu) : (x,y) \mapsto c(x,y) = \mathbb{1}_{H_x} - \mathbb{1}_{H_y}$. Then:

- $\|c(x,y)\|_2^2 = d(x,y)$ (<u>Crofton's formula</u>: the distance between 2 points is, up to a normalizing factor, the measure of the set of hyperplanes separating them).
- if π is the natural rep. of $\mathrm{Isom}\,X_n$ on $L^2(\mathcal{H}_n, \mu)$, then $\pi(g)\,c(x,y) = c(gx, gy)$.

- $c(x,y)+c(y,z)=c(x,z)$ (trivial).

As in the case of trees, set $b(g)=c(gx_0,x_0)$, so $b\in Z^1(\mathrm{Isom}\,X_n,\pi)$

Corollary: If G has property (T), every continuous homom. $\sigma: G\to \mathrm{Isom}(X_n)$ has relatively compact image ($\Leftrightarrow \sigma(G)$ fixes a point in X_n)

Pf: Since $\sigma(G)x_0$ is bounded in X_n, then $C=\overline{\mathrm{conv}\,\alpha(G)x_0}$ is compact, and the barycenter $\frac{1}{m_n(C)}\int_C p\,dm_n(p)$ (where m_n is the invariant volume on X_n) is a fixed point.
($\Leftarrow$ follows from the fact that point stabilizers are maximal compact sbgps in $\mathrm{Isom}(X_n)$).

3. Reduced 1-cohomology.

3.1. Shalom's theorem

Endow $Z^1(G,\pi)$ with the topology of uniform convergence on compact sets. The reduced 1-cohomology is $\overline{H^1}(G,\pi)=: Z^1(G,\pi)/\overline{B^1(G,\pi)}$

(so it is a quotient of $H^1(G,\pi)$). In our dictionary.

α almost has fixed points $\Longleftrightarrow b\in\overline{B^1(G,\pi)}$.

($\forall\varepsilon>0$, $K\subset G$ compact, $\exists v\in\mathcal{H}$: $\max_{g\in K}\|\alpha(g)v-v\|<\varepsilon$)

Clearly: Prop (T) $\Rightarrow \overline{H^1}(G,\pi)=0\ \forall\pi\in\mathrm{Rep}\,G$.

The converse is false however!

Lemma: Let $\chi: G\to U(1)$ be a ~~non trivial~~ character non-trivial on $Z(G)$. Then $H^1(G,\chi)=0$

Pf: Assume $\chi(z_0)\neq 1$, then
$$b(gz_0)=\chi(g)b(z_0)+b(g)$$
$$b(z_0g)=\chi(z_0)b(g)+b(z_0)$$
Re-arranging: $b(g)(1-\chi(z_0))=b(z_0)-\chi(g)b(z_0)$
$$\Rightarrow b(g)=\chi(g)\frac{b(z_0)}{\chi(z_0)-1}-\frac{b(z_0)}{\chi(z_0)-1} \text{ and } b\in B^1(G,\chi).$$ ◻

So, let G be an infinite abelian group with $\mathrm{Hom}(G,\mathbb{C})=0$, e.g. $G=(\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}=\bigoplus_{n=1}^{\infty}\mathbb{Z}/2\mathbb{Z}$. Then $H^1(G,\pi)=0\ \forall\pi\in\widehat{G}$ (the unitary dual of G).

Thm (Guichardet 1971) Let G be σ-compact.
If $\overline{H^1}(G,\pi)=0\ \forall \pi\in\hat{G}$, then $\overline{H^1}(G,\sigma)=0\ \forall\sigma\in \mathrm{Rep}\,G$. ☐

So, for our abelian group $G=\overset{\infty}{\oplus}\mathbb{Z}/2\mathbb{Z}$: $\overline{H^1}(G,\sigma)=0\ \forall\sigma\in\mathrm{Rep}\,G$,
but G does not have (T).

Thm (Y. SHALOM 2001) Suppose G is compactly generated.
G has (T) $\iff \overline{H^1}(G,\pi)=0\ \forall\pi\in\mathrm{Rep}\,G$

$$\left(\underset{\text{Guichardet}}{\iff} \overline{H^1}(G,\pi)=0\quad\forall\pi\in\hat{G}\right).$$

3.2. Sketch of proof, for G discrete

For S finite generating set of G, $\pi\in\mathrm{Rep}\,G$, define
$\delta:\mathcal{H}\to\mathbb{R}^+ : \xi\mapsto\max_{s\in S}\|\pi(s)\xi-\xi\|$

Lemma: Assume that π almost has invariant vectors, but no fixed vector. Then, for $M>1$, $\exists\ \xi_M\in\mathcal{H}$ s.t.
(i) $\delta(\xi_M)=1$
(ii) $\delta(\eta)\geq\frac{1}{2}\ \forall\eta\in B(\xi_M,M)$.

Pf: Enough to prove: $\forall L>1$, $\exists r>0$ and $\zeta_L\in\mathcal{H}$ s.t. $\delta(\zeta_L)=\frac{r}{L}$
and $\delta(\eta)>\frac{r}{2L}\ \forall\eta\in B(\zeta_L,3r)$.
(indeed, then take $\xi_L=\frac{L}{r}\zeta_L$ so $\delta(\xi_L)=1$ and $\delta(\eta)>\frac{1}{2}\ \forall\eta\in B(\xi_L,3L)$
As π almost has invariant vectors, find a unit vector ζ with
with $\delta(\zeta)<\frac{1}{6L}$. Then $\zeta_1=\frac{\zeta}{2\delta(\zeta)L}$
satisfies $\|\zeta_1\|>3$ and $\delta(\zeta_1)=\frac{1}{2L}$.
If $\delta(\eta)>\frac{1}{4L}$ for every $\eta\in B(\zeta_1,\frac{3}{2})$ we choose $r=\frac{1}{2}$
and $\zeta_L=\zeta_1$.
If not, there is $\eta\in B(\zeta_1,\frac{3}{2})$ with $\delta(\eta)\leq\frac{1}{4L}$. By
continuity of δ, find $\zeta_2\in B(\zeta_1,\frac{3}{2})$ with $\delta(\zeta_2)=\frac{1}{4L}$.
If $\delta(\eta)>\frac{1}{8L}$ for every $\eta\in B(\zeta_2,\frac{3}{4})$, take $r=\frac{1}{4}$, $\zeta_L=\zeta_2$

If not, find $\xi_3 \in B(\xi_2, \frac{3}{4})$ with $\delta(\xi_3) = \frac{1}{8}$ ~~and~~.
The process must stop in finitely many steps. Otherwise we construct a sequence $(\xi_n)_{n\geq 1}$ with $\|\xi_{n+1} - \xi_n\| < \frac{3}{2^n}$ and $\delta(\xi_{n+1}) = \frac{1}{2^{n+1}}$.

Then $(\xi_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}$, hence converges to $\xi \in \mathcal{H}$. By continuity of δ: $\delta(\xi) = 0$. By definition of δ: $\pi(s)\xi = \xi$ $\forall s \in S$. Since S generates G, this forces ξ to be $\pi(G)$-fixed, i.e. $\xi = 0$. On the other hand:

$$\|\xi_{n+1} - \xi_1\| \leq \sum_{i=1}^{n} \|\xi_{i+1} - \xi_i\| < 3 \sum_{i=1}^{n} \frac{1}{2^i} < 3$$

$\Rightarrow \|\xi - \xi_1\| \leq 3$. But $\|\xi_1\| > 3$, so $\xi \neq 0$, contrad. $\square$

The idea for the ~~rest~~ conclusion of the proof, is to look at the family of balls $(B(\xi_M, M))_{M>1}$. These are "almost invariant", in the sense that generators in S move ~~by~~ the center ξ_M by at most 1, and they have no almost fixed point, since every point in such a ball is moved by at least $1/2$ by some generator. Using Ascoli-Arzela, Shalom proves that this family of balls converges in the Gromov-Hausdorff distance on metric spaces. The limit space is an affine Hilbert space carrying an isometric action of G, where every point is moved by at least $1/2$ by some generator, so it does not have almost fixed points. Hence the existence of π, $b \in Z^1(G, \pi)$ with $b \in \overline{B^1}(G, \pi)$. $\square$

3.3 The problem of finite presentability

In his original paper, Kazhdan asked: "Is every discrete group with property (T), of finite presentation?".

Consider the group $SL_n(\mathbb{F}_p[X])$. It is isomorphic to a lattice in the ~~lattice~~ algebraic group $SL_n(\mathbb{F}_p((x)))$ over the local field $\mathbb{F}_p((x))$ of Laurent series. In particular it has property (T) for $n \geq 3$ (Kazhdan)

$SL_n(\mathbb{F}_p[X])$	Properties
$n=2$	not finitely generated (Serre 1970)
$n=3$	f.g., not f.p. (Behr 1979)
$n \geq 4$	finitely presented (Rehmann-Soulé 1976)

(negative answer to Kazhdan's question!

However, Shalom (2001) proved

Thm: Let Γ be a discrete Kazhdan group. There exists a finitely presented Kazhdan group G such that $G \twoheadrightarrow \Gamma$.

Open: describe explicitly a f.p. Kazhdan group mapping onto $SL_3(\mathbb{F}_p[X])$.

Assume that Γ is ~~any~~ f.g. group, view Γ as a quotient of some free group $\mathbb{F}_n$, with kernel $N \trianglelefteq \mathbb{F}_n$. Enumerate the elements of N: $N = \{w_1, w_2, \dots\}$. Set $N_k = \langle\langle w_1, \dots, w_k \rangle\rangle$ (normal subgroup generated by $w_1, \dots, w_k$), and $\Gamma_k = \mathbb{F}_n / N_k$, so that $\Gamma_k \twoheadrightarrow \Gamma$ and Γ_k is f.p.

Enough to prove:

Prop: If none of the Γ_k's has (T), then Γ does not have (T).

Pf: Let π_k be a rep of Γ_k almost having invariant vectors, no fixed vector. View it as a rep. of $\mathbb{F}_n$. By the lemma, find $\xi_k \in \mathcal{H}_k$ s.t. $\delta(\xi_k) = 1$ and $\delta(\eta) \geq \frac{1}{2}$ for $\eta \in B(\xi_k, k)$. The sequence of balls $(B(\xi_k, k))_{k \geq 1}$ converges in the Gromov-Hausdorff distance to an affine metric space carrying an action of $\mathbb{F}_n$ without almost fixed points. That action ~~converges~~ is trivial on $\bigcup_{k=1}^{\infty} N_k = N$, so it factors through $\mathbb{F}_n / N = \Gamma$, which therefore does not have (T). $\square$

3.4. Other applications of Shalom's thm:

- Short proof of Serre's thm:

Thm (Serre 1988) Let G be compactly generated, C a closed central subgp. ~~If~~ Assume $G/[G,G]$ is compact. If G/C has (T) then G has (T)

Note: Assumption $G/\overline{[G,G]}$ is clearly necessary: take $G = H \times \mathbb{R}$, H with (T); $G/\mathbb{R}$ has (T) but not G

- Conceptual proof (appealing to ideas of Gromov on harmonic maps) for property (T) for $Sp(n,1)$ and $F_{4(-20)}$. (first proof: Kostant 1969)

4. Expander graphs.

4.1 Motivation

Suppose we want to use finite graphs $X=(V,E)$ to model communication systems where information propagates (say, in 1 time unit) from one vertex to another along edges.

Two constraints:

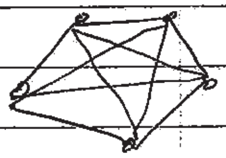
Tension!

(1) Every set of vertices should be "well-connected" to the rest of the vertices, to ensure quick transmission

(2) The number of edges should be kept "small", as transmission lines are made of copper or optical fibers

Moreover: we don't want one graph, but a family of finite graphs, to model arbitrarily large systems

Example: The complete graph K_n is certainly the best in terms of connectedness, but it has $\frac{n(n-1)}{2}$ edges.

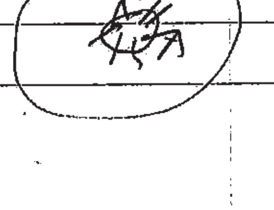


K_5

Consequence: Stick to k-regular graphs (each vertex has k neighbours, k fixed). Advantage: $|E| = \frac{k|V|}{2}$, so number of edges is linear in number of vertices.

To measure connectivity: For $A \subset V$, let ∂A be the boundary of A: $\partial A = \{e \in E : e \text{ connects } A \text{ to } V \setminus A\}$

We want $|\partial A|$ to be large with respect to $|A|$. Observe that $\partial A = \partial(V \setminus A)$, so we restrict to subsets A with $|A| \leq \frac{1}{2}|V|$.



X

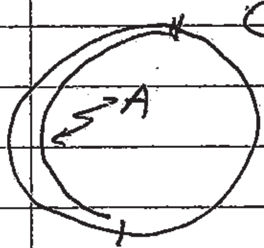
Define the <u>isoperimetric constant</u> of a graph X (finite, connected) as

$$h(X) = \min_{0<|A|\leq \frac{|V|}{2}} \frac{|\partial A|}{|A|}.$$

Def. A family $(X_n)_{n\geq 1}$ of finite, connected, k-regular graphs is a <u>family of expanders</u> if $\exists\, \varepsilon > 0$ s.t. $h(X_n) \geq \varepsilon \quad \forall n \geq 1$, and $\lim_{n\to\infty} |X_n| = +\infty$.

<u>Non-example</u>: $k=2$. Cycles do <u>not</u> form an expander family! If A is a half-cycle, then $|\partial A| = 2$

C_n: $\Rightarrow h(C_n) = O\left(\frac{1}{n}\right)$.



In 1973, Pinsker proved existence of expanders by a counting argument: For $\varepsilon > 0$ given, the proportion of k-regular graphs on vertices with $h(X) < \varepsilon$, goes to 0 for $n \to \infty$.

<u>4.2. Margulis' construction.</u>

If G is a finitely generated group and S is a finite symmetric generating subset, then the Cayley graph $\mathcal{G}(G,S)$ is the graph with $V = G$ and $E = \{(g, gs) : g \in G, s \in S\}$. It is k-regular with $k = |S|$, connected, and vertex-transitive (G acts on the left).

If $X = (V,E)$ is connected finite, k-regular, the <u>Laplace operator</u> $\Delta : \ell^2(V) \to \ell^2(V)$: $\Delta f(x) = k f(x) - \sum_{y\sim x} f(y)$.

It is a non-negative operator, with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{n-1}$

If we orient the edges $\underset{e^-}{\circ}\!\xrightarrow{\;e\;}\!\underset{e^+}{\circ}$, we get the coboundary operator $d : \ell^2(V) \to \ell^2(E) : df(e) = f(e_+) - f(e_-)$. Then $d^*\xi(v) = \sum_{e : e_+ = v} \xi(e) - \sum_{e : e_- = v} \xi(e)$

and $d^* d f(x) = \sum_{e : e_+ = x} df(e) - \sum_{e : e_- = x} df(e)$

$= \sum_{e_+ = x} (f(x) - f(e_-)) - \sum_{e_- = x} (f(e_+) - f(x))$

$= k f(x) - \sum_{y \sim x} f(y) = \Delta f(x)$.

So, by the Rayleigh principle:

$\lambda_1 = \inf \{ \frac{\langle \Delta f | f \rangle}{\|f\|^2} : f \in \ell^2(V), f \perp 1, f \neq 0 \}$

$= \inf \{ \frac{\|df\|^2}{\|f\|^2} : f \perp 1, f \neq 0 \}$.

Lemma: $\lambda_1 \leq 2 h(X)$.

Pf: Let $A \subset V$, $|A| \leq \frac{|V|}{2}$, s.t. $h(X) = \frac{|\partial A|}{|A|}$.

Set $\xi_A(x) = \begin{cases} |V \setminus A| & \text{if } x \in A \\ -|A| & \text{if } x \in V \setminus A. \end{cases}$

Then $\xi_A \perp 1$, $\|\xi_A\|_2^2 = |A| |V \setminus A|^2 + |V \setminus A| |A|^2$
$= |A| |V \setminus A| n$.

While $d\xi_A(e) = \begin{cases} 0 & \text{if } e \notin \partial A \\ \pm n & \text{if } e \in \partial A \end{cases}$

$\Rightarrow \|d\xi_A\|_2^2 = n^2 |\partial A|$. So $\lambda_1 \leq \frac{\|d\xi_A\|^2}{\|\xi_A\|^2} = \frac{n^2 |\partial A|}{n |A| |V \setminus A|}$

But $|V \setminus A| \geq \frac{n}{2}$, so $\lambda_1 \leq \frac{2 |\partial A|}{|A|} = 2 h(X)$. ☐

Now for a Cayley graph: $\Delta f(x) = |S| f(x) - \sum_{s \in S} f(xs)$
$= |S| f(x) - \sum_{s \in S} (\rho(s) f)(x)$
or $\Delta = |S| \, Id - \sum_{s \in S} \rho(s)$

Thm (Margulis 1973): Let G be a discrete, infinite group with property (T). Assume that G is residually finite, i.e. $\exists N_n \triangleleft G$, $[G : N_n] < +\infty$, $N_{n+1} \subset N_n$, s.t. $\bigcap_{n=1}^{\infty} N_n = \{1\}$

Set $G_n = G/N_n$

[Then for every finite symmetric generating set S of G: the family $\left(\mathcal{G}(G_n, S)\right)_{n\geq 1}$ is a family of expanders

Example: $G = SL_N(\mathbb{Z})$, $(N \geq 3)$, $G_n = SL_N(\mathbb{Z}/n\mathbb{Z})$.

Pf: By property (T), $\exists \varepsilon > 0$: $\inf_{\|\xi\|=1} \max_{s\in S} \|\pi(s)\xi - \xi\| < \varepsilon$ $\Rightarrow$ π has invariant vectors.

Let π_n be the rep of G on $\ell^2_0(G_n)$ obtained by composing the quotient map $G \to G_n$ with the right rep ρ_n of G_n. Since π_n has no invariant vector, $\forall \xi \in \ell^2_0(G_n)$, $\|\xi\|=1, \exists s_0 \in S$: $\|\pi(s_0)\xi - \xi\| \geq \varepsilon$; squaring

$$2 - 2\,\mathrm{Re}\langle \pi(s_0)\xi|\xi\rangle \geq \varepsilon^2$$

$$\Rightarrow 1 - \frac{\varepsilon^2}{2} \geq \mathrm{Re}\langle \pi(s_0)\xi|\xi\rangle.$$

So $\langle \Delta\xi|\xi\rangle = |S| - \sum_{s\in S}\langle \pi_n(s)\xi|\xi\rangle$

$$= |S| - \Big(\sum_{\substack{s\in S\\ s\neq s_0}} \mathrm{Re}\langle \pi_n(s)\xi|\xi\rangle\Big) - \mathrm{Re}\langle \pi_n(s_0)\xi|\xi\rangle$$

$$\geq |S| - (|S|-1) - \left(1 - \frac{\varepsilon^2}{2}\right) = \frac{\varepsilon^2}{2}.$$

Hence $\lambda_1(\mathcal{G}(G_n, S)) \geq \frac{\varepsilon^2}{2}$, proof completed ☐