# 230b1206
# The Clifford algebra.

February 9-14, 2012

The goal of the next few lectures is to go through Cartan's study of the spin representation of even orthogonal groups in an amazing paper in (1913!) in *Bull. Soc. Math. France* 41 53-96.

A "modern" version of his results appear in the book *The algebraic theory of spinors* by C. Chevalley in 1954. Dieudonné's review of this book in the Bull. AMS (1954) is particularly enlightening.

I have posted on the web site of this course Meinrenken's recent notes, which are very relevant to what we will be doing.

# Contents

# The definition of the Clifford algebra.

$V$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a non-degenerate symmetric bilinear form $B(\cdot,\cdot)$. We will sometimes refer to such a bilinear form $B$ as an inner product or a scalar product on $V$ . The **Clifford algebra** $Cl(V)$ over $V$ is the associative unital algebra, generated by the elements of $V$, with relations,

$$vv' + v'v = B(v, v')1.$$

This is the convention used by the algebraists, most particularly the one used by Chevalley to allow a good algebra over fields of characteristic 2, and is the convention that we will use

Since we are over the real (mainly) or complex numbers, we can write the relations as

$$v^2 = \frac{1}{2}B(v, v).$$

# Construction of the Clifford algebra.

We can construct $Cl(V)$ abstractly as the quotient of the tensor algebra $T(V)$ by the two sided ideal generated by the elements $v \otimes v' + v' \otimes v - B(v, v')1$. This then allows $Cl(V)$ to be characterized as the solution to a universal problem, see below. In particular, if $V$ is isometric to $W$ then $Cl(V)$ is isomorphic to $Cl(W)$.

So if, $B$ is non-degenerate, and we are over the real numbers, then $Cl(V)$ is determined, up to isomorphism, by the integers $(p, q)$, the number $p$ of $+$ signs and the number $q$ of negative signs in the Sylvester normal form of $B$. So we can denote the Clifford algebra by $C(p, q)$. We will be almost exclusively interested in the split case where $p = q$. But I will begin with some general considerations.

# The gradation, the filtration, and the anti-automorphism.

$Cl(V)$ is $\mathbb{Z}_2$ graded and has a $\mathbb{Z}$ filtration both inherited from the tensor algebra. The generators $v \in V$ are odd and have filtration degree 1.

$$x \mapsto x^T$$

will denote the canonical anti-automorphism which equals the identity on $V$. So for elements of $V$ we have

$$(x_1 x_2 \cdots x_k)^T = x_k x_{k-1} \cdots x_2 x_1.$$

It is an anti-automorphism in the sense that for any $u, v \in Cl(V)$ we have

$$(uv)^T = v^T u^T,$$

and $(u^T)^T = u$.

I will also use the notation $x \mapsto x^T$ to denote the anti-automorphism of the exterior algebra (which can be thought of as a Clifford algebra with $B \equiv 0$). So

$$(x_1 \wedge \cdots \wedge x_k)^T = x_k \wedge \cdots \wedge x_1.$$

So in the exterior algebra we have, for $x_i \in V$

$$\begin{aligned} x_1^T &= x_1 \\ (x_1 \wedge x_2)^T = x_2 \wedge x_1 &= -(x_1 \wedge x_2) \\ (x_1 \wedge x_2 \wedge x_3)^T &= -(x_1 \wedge x_2 \wedge x_3) \\ (x_1 \wedge x_2 \wedge x_3 \wedge x_4)^T &= (x_1 \wedge x_2 \wedge x_3 \wedge x_4) \end{aligned}$$

and, in general,

in the exterior algebra,

$$(x_1 \wedge \cdots \wedge x_m)^T = (-1)^{m(m-1)/2}(x_1 \wedge \cdots \wedge x_m),$$

since it takes $(m-1)$ transpositions to move the $x_m$ to the first position, then $(m-2)$ transpositions to move the $x_{m-1}$ to the second position, etc.
In more prosaic terms

$$(x_1 \wedge \cdots \wedge x_m)^T = (x_1 \wedge \cdots \wedge x_m)$$

if $m \equiv 0$ or $1 \bmod 4$, and

$$(x_1 \wedge \cdots \wedge x_m)^T = -(x_1 \wedge \cdots \wedge x_m)$$

if $m \equiv 2$ or $3 \bmod 4$.

# Clifford maps and Clifford modules.

A linear map $f$ of $V$ to an associative algebra $A$ with unit $1_A$ is called a **Clifford map** if

$$f(y_1)f(y_2) + f(y_2)f(y_1) = B(y_1, y_2)1_A, \quad \forall y_1,\ y_2\ \in V$$

or what amounts to the same thing (by polarization since we are not over a field of characteristic 2) if

$$f(y)^2 = \frac{1}{2}(y, y)1_A \ \ \forall\ y \in V.$$

Any Clifford map gives rise to a unique algebra homomorphism of $Cl(V)$ to $A$ whose restriction to $V$ is $f$. The Clifford algebra is "universal" with respect to this property.

In particular, a vector space $M$ is called a Clifford module if we are given a Clifford map from $V$ to $\mathrm{End}(M)$. If $\varrho : Cl(V) \to \mathrm{End}(M)$ gives the action of $Cl(V)$ on the module $M$, then the dual space $M^*$ becomes a Clifford module under the action

$$\varrho^*(x) := \varrho(x^T)^*.$$

# Interior product and exterior product.

Since $V$ has a non-degenerate "scalar product", we have an identification of $V$ with $V^*$. For $v \in V$, we will let $i(v)$ denote the derivation of $\wedge V$ corresponding to interior product by the image of $v$ in $V^*$. Explicitly, if $v \neq 0$, we can extend $v$ to a basis of $V$ and then write every element of $\wedge V$ as

$$v \wedge \sigma + \omega$$

where $\sigma$ and $\omega$ do not have any factor of $v$ in their expression. Then

$$i(v)[v \wedge \sigma + \omega] = B(v, v)\sigma.$$

Let $\epsilon(v)$ denote exterior multiplication by $v$ on $\wedge(V)$.

So $\quad \epsilon(v)[v \wedge \sigma + \omega] = v \wedge \omega \quad$ and so

$$i(v)\epsilon(v)[v \wedge \sigma + \omega] = B(v, v)\omega$$

while

$$\epsilon(v)i(v)[v \wedge \sigma + \omega] = B(v, v)v \wedge \sigma.$$

Thus

$$(i(v) + \epsilon(v))^2 = \epsilon(v)i(v) + i(v)\epsilon(v) = B(v, v)\text{Identity}$$

acting on $\wedge(V)$.
In our convention, this is the Clifford identity for $2B$.

# The symbol map.

So let us define

$$\rho(v) := \epsilon(v) + \frac{1}{2} i(v).$$

Then $\rho(v)^2 = \frac{1}{2} B(v, v) \cdot \text{Identity}$ and so extends to an action (call it $\rho$, also) of $Cl(V)$ on $\wedge(V)$.

**Definition**

The **symbol map** $\sigma : Cl(V) \to \wedge(V)$ is defined by

$$\sigma(x) = \rho(x)(1)$$

where the 1 on the right hand side is the number $1 \in \mathbb{K} = \wedge^0(V)$.

# The symbol map is an isomorphism of filtered super vector spaces.

**Proof.**

Let $I = (i_1, \ldots, i_k),\ 1 \leq i_1 < i_2 < \cdots < i_k \leq n = \dim V$ denote an ordered subset and

$$e_I := e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_k}$$

where the $e_i$ form an "orthogonal" basis of $V$. The $e_I$, as $I$ ranges over all ordered subsets form a basis of $Cl(V)$. In computing $\rho(e_I)(1)$, all the interior products vanish, and hence

$$\sigma(e_I) = e_{i_1} \wedge \cdots \wedge e_{i_k}$$

and these form a basis of $\wedge(V)$. $\square$

# The symbol map in low degrees.

For degree $\leq 3$ these are

$$\begin{aligned}
\sigma(1) &= 1 \\
\sigma(v) &= v \\
\sigma(v_1 v_2) &= v_1 \wedge v_2 + \frac{1}{2} B(v_1, v_2) \\
\sigma(v_1 v_2 v_3) &= v_1 \wedge v_2 \wedge v_3 + \frac{1}{2}[B(v_2, v_3)v_1 - B(v_1, v_3)v_2 + B(v_1, v_2)v_3].
\end{aligned}$$

## The symbol map induces an isomorphism of graded superalgebras $\mathrm{gr}\, Cl(V) \to \wedge(V)$.

Indeed, we know that that the symbol map is an isomorphism of filtered super spaces, the associated graded map is an isomorphism of graded super spaces. To check that the induced map preserves products, we must show that the symbol map intertwines products up to lower order terms. But this is clear from the definition.

# The quantization map.

This map

$$q : \wedge(V) \to Cl(V)$$

is defined as the inverse of the symbol map. So

$$q(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_I.$$

Here is an instructive alternative definition of the quantization map:

The quantization map is given by super symmetrization. That is, for $v_1, ..., v_k \in V$,

$$q(v_1 \wedge ... \wedge v_k) = \frac{1}{k!} \sum_{s \in \mathfrak{S}} \mathrm{sign}(s) v_{s(1)} \cdots v_{s(k)}$$

where $\mathfrak{S}$ denotes the group of permutations of $1, \ldots, k$ and $\mathrm{sign}(s) = \pm 1$ is the parity of the permutation $s$.

# Proof of the alternative definition of the quantization map.

By linearity, it suffices to check for the case that the $v_j$ are elements of an orthogonal basis $e_1, ..., e_n$ of $V$ That is, $v_j = e_{i_j}$ where the indices $i_j$ need not be ordered or distinct). If the $i_j$ are all distinct, then the $e_{i_j}$ Clifford (super) commute , and the right hand side coincides with the left hand side. If any two $e_{i_j}$ are equal, then both sides are zero. $\square$

As an immediate corollary to the alternative definition we have,

**Corollary**

*If $u \in \wedge V$ then*

$$q(u^T) = q(u)^T.$$

*Here the $^T$ on the right is the transposition in the Clifford algebra while the $^T$ on the left is in the exterior algebra.*

# A corollary to the corollary.

## Corollary

*For any $\phi \in \wedge^k(V)$ we have*

$$q(\phi)^2 \in q\left(\wedge^0(V) \oplus \wedge^4(V) \oplus \cdots \oplus \wedge^{4r}(V)\right)$$

*where $r$ is the largest integer with $2r \leq k$.*

## Proof.

The element $q(\phi)^2$ is even and is of filtration degree at most $2k$. So to prove the corollary, we must show that elements of $\wedge^m(V)$ with $m \equiv 2 \bmod 4$ make no contribution.

Now $q(\phi)^T = q(\phi^T) = \pm q(\phi)$ since $\phi$ is homogeneous of degree $k$. So $(q(\phi)^2)^T = (q(\phi)^T)^2 = q(\phi)^2$. So if we let $\psi = \sigma(q(\phi)^2)$ so that $q(\phi)^2 = q(\psi)$, we must have $\psi^T = \psi$, so $\psi$ can have no components lying in $\wedge^m V$ where $m \equiv 2 \bmod 4$.

$\square$

# The (super) tensor product of two superalgebras .

If $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ are superalgebras, i.e. $\mathbb{Z}_2$ graded algebras, then their tensor product $A \otimes B$ is the superalgebra with

$$(A \otimes B)_0 = A_0 \otimes B_0 \ \oplus \ A_1 \otimes B_1,$$

$$(A \otimes B)_1 = A_0 \otimes B_1 \ \oplus \ A_1 \otimes B_0$$

and with multiplication

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb',$$

where $|\ |$ denotes degree. This is the convention we will use from now on, in contrast to the notion of tensor product in the statement of Bott periodicity.

For example, let $X = X_0 \oplus X_1$ be a super vector space, and let $\mathcal{A} = \mathrm{End}(X)$ so that $\mathcal{A}$ is a superalgebra whose even elements preserve degree and whose odd elements reverse degree. Similarly, let $\mathcal{B} = \mathrm{End}(Y)$ where $Y$ is a supervector space. We let $\mathcal{A} \otimes \mathcal{B}$ act on $X \otimes Y$ by

$$(A \otimes B)(x \otimes y) = (-1)^{|B||x|} Ax \otimes By$$

for homogeneous elements. Then it is routine to check that

$$\mathrm{End}(X \otimes Y) \sim \mathrm{End}(X) \otimes \mathrm{End}(Y).$$

# The Clifford algebra of direct sum.

Let $E$ and $F$ be vector spaces with scalar products, and

$$H = E \oplus F$$

their direct sum with the direct sum scalar product (so $E$ and $F$ are orthogonal subspaces of $H$). Let $i : E \to H,\ j : F \to H$ denote the inclusion maps. Since they are isometric injections, they induce homomorphisms (which we will also denote by $i$ and $j$) of the Clifford algebras:

$$i : Cl(E) \to Cl(H), \quad j : Cl(F) \to Cl(H).$$

The elements $i(e),\ e \in E$ and $j(f),\ f \in F$ are orthogonal, and hence

$$i(e)j(f) + j(f)i(e) = 0$$

and so $k := i \otimes j$ is an algebra homomorphism

$$k : Cl(E) \otimes Cl(F) \to Cl(H).$$

To show that $k$ is an isomorphism, we construct the inverse: define

$$\kappa : H \to Cl(E) \otimes Cl(F), \quad \kappa(e \oplus f) = e \otimes 1_F + 1_E \otimes f$$

where $1_E$ and $1_F$ are the identity elements of $Cl(E)$ and $Cl(F)$. Then

$$\begin{aligned}(\kappa(e \oplus f))^2 &= (B_E(e,e) + B_F(f,f))1_E \otimes 1_F \\ &= B_H(e \oplus f, e \oplus f)1_{Cl(E)\otimes Cl(F)}\end{aligned}$$

and so $\kappa$ extends to an algebra homomorphism (which we also denote by $\kappa$)

$$\kappa : Cl(H) \to Cl(E) \otimes Cl(F).$$

It is then easy to check that $\kappa \circ k$ and $k \circ \kappa$ are the identity on generators, and so are mutual inverses.

In short, we have established an isomorphism

$$Cl(E \oplus F) \sim Cl(E) \otimes Cl(F).$$

# The Clifford algebra of a split space.

Suppose that $\mathbb{V}$ is a(n even dimensional) vector space of signature (m,m). We can decompose $\mathbb{V}$ into a direct sum of two dimensional split spaces

$$\mathbb{V} = E_1 \oplus E_2 \oplus \cdots \oplus E_m$$

and each $Cl(E_i)$ is isomorphic to $\mathrm{End}(\wedge T_i)$ where $T_i$ is a one dimensional vector space. If we let

$$T = T_1 \oplus T_2 \oplus \cdots \oplus T_m$$

then

$$\wedge T \sim \wedge T_1 \otimes \cdots \otimes \wedge T_m$$

and hence

$$Cl(\mathbb{V}) \sim \mathrm{End}(\wedge T).$$

In particular, by Wedderburn's theorem, $Cl(\mathbb{V})$ has a unique irreducible module and any module is a direct sum of copies of this module. We will prove a stronger version of these facts more directly.

# Left and right multiplication, commutator.

For any $x \in Cl(X)$ the operation $\ell^{Cl}(x)$ will denote left multiplication by $x$ so

$$\ell^{Cl}(x)x' := xx'$$

while $r^{Cl}(x)$ will denote right multiplication by $x$ with the appropriate sign, so

$$r^{Cl}(x)x' = (-1)^{|x||x'|}x'x.$$

With this notation, $\ell^{Cl}(x) - r^{Cl}(x)$ is the operator of left super commutator

$$[x, \cdot]_{Cl}.$$

# The trace.

A (super)**trace** on a superalgebra is a linear function which vanishes on (super)commutators. We show that the space of supertraces of the Clifford algebra is one dimensional: To prove this, we show that

$$[Cl(V), Cl(V)] = Cl^{n-1}(V)$$

where $n = \dim(V)$ and $Cl^{n-1}(V)$ denotes the subset of $Cl(V)$ of filtration degree $n-1$, that is the set of elements of $Cl(V)$ which are sums of terms which are products of at most $n-1$ factors of elements of $V$. To see this, let $v_1, \ldots, v_n$ be an orthogonal basis of $V$ so that the

$$v_I := v_{i_1} v_{i_2} \cdots v_{i_k}, \quad I = (i_1, \ldots, i_k)$$

form a basis of $Cl(V)$ as $I$ ranges over all (ordered) subsets of $\{1, \ldots, n\}$.

We first show that

$$[v_I, v_J] = v_I v_J - (-1)^{|I|+|J|} v_J v_I \in Cl^{n-1}(V).$$

If $I \cap J = \emptyset$ then this commutator vanishes by the Clifford relations.. So suppose that $I$ and $J$ have an element in common. Without loss of generality, we may assume that this element is $v_1$. So we want to consider

$$[v_1 v_K, v_1 v_L]$$

where $K$ and $L$ are subsets of $\{2, \ldots, n\}$.

But

$$v_1 v_K v_1 v_L = (-1)^{|K|} v_1^2 v_K v_L, \qquad v_1 v_L v_1 v_K = (-1)^{|L|} v_1^2 v_L v_K,$$

and $v_1^2 = \frac{1}{2} B(v_1, v_1) 1$ is a scalar. So $[v_1 v_K, v_1 v_L]$ involves only products of $v_2, \ldots, v_n$ and so lies in $Cl^{n-1}(V)$. We have shown that

$$[Cl(V), Cl(V)] \subset Cl^{n-1}(V).$$

To prove equality, it is enough to show that every element $v_I$ with $|I| < n$ can be expressed as a commutator. Since $|I| < n$ there is some $i \notin I$. Without loss of generality we may assume that $i = 1$ so $I \subset \{2, \ldots, n\}$. But

$$[v_1, v_1 v_I] = B(v_1, v_1) v_I$$

and $B(v_1, v_1) \neq 0$. This proves that

$$[Cl(V), Cl(V)] = Cl^{n-1}(V).$$

Since $q^{-1}$ induces an isomorphism

$$q^{-1}: \; Cl(V)/Cl^{n-1}(V) \to \wedge^n(V)$$

we can choose as our trace the map $\mathrm{tr}_s$ to be $q^{-1}$ followed by the projection of $\wedge(V)$ onto $\wedge^n(V)$. (This becomes a scalar valued function once we choose an orientation on $V$ which, together with the metric $B$ gives a basis of $\wedge^n(V)$.)

# Recall.

Let $A$ be a filtered superalgebra whose even elements are filtered by even integers and whose odd elements are filtered by odd integers. Suppose that the associated graded algebra $\operatorname{gr} A$ is (super) commutative. Then $\operatorname{gr} A$ inherits a Poisson bracket defined as follows: If $x \in A^k, y \in A^\ell$ then the supercommutativity of $\operatorname{gr} A$ implies that $[x, y] \in A^{k+\ell-2}$, if $x$ and $y$ are of definite parity. So if

$$X = \operatorname{gr} x, \quad Y = \operatorname{gr} y$$

we define

$$\{X, Y\} := \operatorname{gr}[x, y].$$

It is easy to check that this is well defined, and that the usual axioms for (super) Poisson bracket are satisfied: that $\{\cdot, \cdot\}$ satsifies the axioms for a Lie superalgebra, and that left bracket is a (super) derivation of the (super) commutative multiplication.

For example, if $A$ is the ring of (scalar) differentiable operators on a manifold $M$ with smooth coefficients, (and $A = A_0, A_1 = \{0\}$ and we filter by double the degree) then $\mathrm{gr}\, A$ is the collection of smooth functions on $T^*M$ which are polynomials in the cotangent variables, and the Poisson bracket is the usual one for functions on $T^*M$.

In our case, $\mathrm{gr}\, Cl(V) = \wedge(V)$ so $\wedge(V)$ has a Poisson bracket.

# Bracket by an element of $V$.

For $x \in V$ we may identify $X = \mathrm{gr}\, x$ with $x$. Poisson bracket by $x$ must reduce degree by one. I claim that

$$\{x, w\} = \iota(x)w \tag{1}$$

where $\iota(x)$ denotes interior product by the element of $V^*$ associated to $x$ by $B$. Indeed, $\{x, \cdot\}$ and $\iota(x)$ are both derivations of $\wedge(V)$. So it suffices to verify (1) on generators: but for $y \in V$ we have

$$[x, y] = xy + yx = B(x, y) = \iota(x)y. \quad \square$$

As a corollary we conclude that

# $Cl(V)$ has no proper two sided ideals.

Indeed, any two sided ideal would have to be invariant under commutator bracket by all elements of $V$. So if not zero, repeated bracket would yield a non-zero scalar in the ideal which would then have to be all of $Cl(V)$.
As a corollary to this corollary we see that

**Corollary**

*If $\rho : Cl(V) \to A$ is a non-zero Clifford map, then it must be injective.*

Indeed, its kernel is a two sided ideal.

# Bracket by an element of $\wedge^2(V)$.

Suppose that

$$u \in \wedge^2 V.$$

Then for $y \in V$ we have

$$[u, y] = -[y, u] = -\iota(y)u. \tag{2}$$

In particular, if $u = y_i \wedge y_j$ where $y_i, y_j \in V$ we have

$$[u, y] = (y_j, y)y_i - (y_i, y)y_j \quad \forall\ y \in V. \tag{3}$$

If $(y_i, y_j) = 0$ this is an "infinitesimal rotation" in the plane spanned by $y_i$ and $y_j$. Since $y_i \wedge y_j,\ i < j$ form a basis of $\wedge^2 V$ if $y_1, \ldots, y_n$ form an "orthonormal" basis of $V$, we see that the map

$$u \mapsto [u, \cdot]$$

gives an isomorphism of $\wedge^2 V$ with the orthogonal algebra $o(V)$. Under this identification, Poisson bracket by an element of $\wedge^2(V)$ corresponds to the derivation action of $o(V)$ on $\wedge(V)$ induced from its action on $V$.

# The derivations associated to an element of $\wedge^2(V)$.

We have seen that the action by bracket of an element $\omega$ of degree 2 acting on $V$ gives an isomorphism of $\wedge^2(V)$ with the orthogonal algebra $o(V)$. The action of the group $O(V)$ on $V$ (by isometries) induces a corresponding action by automorphisms on $Cl(V)$ and on $\wedge(V)$. So

- Bracketing via commutator by $\omega$ on $Cl(V)$ gives the derivation, call it $X(\omega)$, of the corresponding element of $o(V)$ coming from the action of $O(V)$ as automorphisms of $Cl(V)$.
- Bracketing via Poisson bracket by $\omega$ on $\wedge(V)$ gives the derivation, call it $Y(\omega)$, of corresponding element of $o(V)$ coming from the action of $O(V)$ as automorphisms of $\wedge(V)$.

# Lie algebras with invariant, non-degenerate bilinear form.

I interrupt the discussion of Cartan's 1913 paper with a result from Kostant and Sternberg *Ann. Phy.* (1987) which will be important to us later. This involves a distinguished element of degree 3 in the Clifford algebra of a Lie algebra with invariant scalar product.
Let $\mathfrak{g}$ be a Lie algebra and suppose that $(\ ,\ )$is a nondegenerate scalar product on $\mathfrak{g}$. Invariance means that

$$([u, v], w) + (v, [u, w]) = 0 \quad \forall u, v, w \in \mathfrak{g}.$$

We are *not* assuming that $(\ ,\ )$ is positive definite, only that it is non-degenerate.

For example, if $\mathfrak{g}$ is a semisimple Lie algebra then the Killing form is nondegenerate and invariant.

# The semi-direct product of $\mathfrak{g}$ and $\mathfrak{g}^*$ as an example.

Here is another class of examples which is important for us. Let $\mathfrak{g}$ be an arbitrary Lie algebra, and let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. Make the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ into a Lie algebra as follows:

The bracket of two elements of $\mathfrak{g}$ is the element of $\mathfrak{g}$ given by the original Lie algebra structure on $\mathfrak{g}$. The bracket of two elements of $\mathfrak{g}^*$ is zero. If $x \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$ then define $[x, f] \in \mathfrak{g}^*$ by the formula

$$[x, f](y) = -f([x, y]) \tag{$*$}$$

and, of course, $[f, x] = -[x, f]$.

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$$[x, f](y) = -f([x, y]) \tag{$*$}$$

and, of course, $[f, x] = -[x, f]$. Define the scalar product on $\mathfrak{g} \oplus \mathfrak{g}^*$ by $(x, y) = 0 = (f, h), \quad x, y \in \mathfrak{g}, \ f, g \in \mathfrak{g}^*$ and

$$(x, f) = (f, x) = f(x), \ x \in \mathfrak{g}, \ f \in \mathfrak{g}^*. \tag{$**$}$$

It is easy to check that this scalar product is invariant.

The set of all finite-dimensional Lie algebras with nondegenerate scalar product has been classified by Medina and Revoy. Basically, what they show is that the most general irreducible such object can be built up from a simple algebra and from an algebra which is a slight generalization of the $\mathfrak{g} \oplus \mathfrak{g}^*$ example constructed above.

# The Cartan three form.

Let $\mathfrak{g}$ be a Lie algebra with invariant scalar product $(\ ,\ )$. Define the trilinear form $\Omega$ by

$$\Omega(x, y, z) = -\left(x, [y, z]\right).$$

The antisymmetry of the Lie bracket together with the invariance of the scalar product imply that $\Omega$ is completely antisymmetric in $x, y$ and $z$. Now since $\mathfrak{g}$ is finite-dimensional (as we assume throughout this course) and $(\ ,\ )$ is non- degenerate, we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and, similarly, we can use the scalar product to identify the space of antisymmetric trilinear forms with $\wedge^3\mathfrak{g}$. Thus we may regard

$$\Omega \in \wedge^3(\mathfrak{g}).$$

We shall call $\Omega$ the **Cartan three form**.

For any Lie algebra $\mathfrak{g}$, if $\ell \in \mathfrak{g}^*$ is a linear function on $\mathfrak{g}$, then the bilinear function $d\ell$ is defined by

$$d\ell(y, z) := -\ell([y, z]).$$

The scalar product on $\mathfrak{g}$ identifies $\mathfrak{g}$ with $\mathfrak{g}^*$ and hence we can think of $d$ as a map from $\mathfrak{g}$ to $\wedge^2\mathfrak{g}$. Thus

$$dx(y, z) = -(x, [y, z]) = (i(x)\Omega)(y, z).$$

We know that interior product by any element of $\mathfrak{g}$ acting on $\wedge\mathfrak{g}$ is the same as Poisson bracket by $x$. So since both $x$ and $\Omega$ are odd, we know that

$$\{\Omega, x\} = \{x, \Omega\} = dx, \qquad \forall x \in \mathfrak{g}.$$

# $\{\Omega, \cdot\} = d.$

Now both Poisson bracket by $\Omega$ and $d$ are superderivations of $\wedge(\mathfrak{g})$ and are therefore determined by their action on $\mathfrak{g}$. Hence

$$\{\Omega, \cdot\} = d.$$

We have thus represented the operator $d : \wedge^k \mathfrak{g} \to \wedge^{k+1} \mathfrak{g}$ as a Poisson bracket by $\Omega$.

Since $d$ and the metric determines the Lie bracket, this means that we can recover the original Lie bracket on $\mathfrak{g}$ from $\Omega$ and the metric.

# The Lie bracket on $\mathfrak{g}$ in terms of $\Omega$ and the metric.

Let me be more explicit: For $x \in \mathfrak{g}$ we have $\{x, \Omega\} = \{\Omega, x\}$ is the element of $\wedge^2\mathfrak{g}$ given by

$$\{x, \Omega\}(y, z) = (i(x)\Omega)(y, z) = -\langle x, [y, z]_{\mathfrak{g}}\rangle.$$

So

$$\{\{\Omega, x\}, y\}(z) = -\{y, \{\Omega, x\}\} = \langle x, [y, z]_{\mathfrak{g}}\rangle = \langle [x, y]_{\mathfrak{g}}, z\rangle.$$

Thus if we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the metric, we see that

$$[x, y]_{\mathfrak{g}} = \{\{\Omega, x\}, y\}.$$

This formula and its generalizations will be of immense importance for us later in the course. Basically, we will be using the analogue of the right hand side to try to define the bracket on the left.

Let $Q \in Cl_3^1(\mathfrak{g})$ with $\mathrm{gr}\, Q = \Omega$. So $Q$ is any odd element of our Clifford algebra with filtration degree 3, whose equivalence class modulo the elements of filtration degree one is $\Omega$.

We know that $Q^2$ is of filtration degree at most 6. In fact, we will prove that $Q^2$ is of filtration degree at most 2, and that by appropriate choice of the representative $Q$, we can arrange that $Q^2$ is of filtration degree zero, i.e. is a scalar.

It turns out that this result of ours from 1987 is a key ingredient (or at least a key motivation) for much of the modern developments as we shall see.

# $Q^2$ is of filtration degree at most 2.

Let us first prove that $Q^2$ is of filtration degree at most 4. Indeed, $\mathrm{gr}_6\, Q^2 = \Omega \wedge \Omega = 0$ since $\Omega \in \wedge^3 \mathfrak{g}$ is of odd degree.

Let $\sigma = \mathrm{gr}_4\, Q^2$. We want to show that $\sigma = 0$, or what is the same thing, that $i(x)\sigma = 0 \quad \forall x \in \mathfrak{g}$. Now

$$i(x)\sigma = \{x, \sigma\} = -\{\sigma, x\} = \mathrm{gr}_3[Q^2, x] = \mathrm{gr}_3[Q, [Q, x]].$$

Let me expand on the last of these equations. Since $Q$ and $x$ are both odd, $[Q, x] = Qx + xQ$ which is even. So $[Q, [Q, x]] = Q(Qx + xQ) - (Qx + xQ)Q = [Q^2, x]$.

Now $\mathrm{gr}_2[Q, x] = \{\Omega, x\} = dx$. So

$$\mathrm{gr}_3[Q, [Q, x]] = \{\Omega, dx\} = d(dx) = 0. \qquad \square$$

Or, more abstractly, we have

$$\{\Omega, \Omega\} = \mathrm{gr}_4[Q, Q] = 2\,\mathrm{gr}_4\, Q^2 = 0.$$

So, since $\Omega$ is odd,

$$\{\{x, \{\Omega\}, \Omega\} = \{x, \{\Omega, \Omega\} - \{x, \Omega\}, \Omega\} = -\{\{x, \Omega\}, \Omega\}$$

so $\{\{x, \Omega\}, \Omega\} = 0$.

We will encounter this type of argument frequently in the future.

Let $c := \mathrm{gr}_2\, Q^2 \in \wedge^2 \mathfrak{g}$. I claim that

$$dc = \{\Omega, c\} = 0.$$

indeed, $\{\Omega, c\} = \mathrm{gr}_3[Q, Q^2]$ since $c = \mathrm{gr}_2\, Q^2$ and $\Omega = \mathrm{gr}_3\, Q$. But

$$[Q, Q^2] = Q \cdot Q^2 - Q^2 \cdot Q = 0.$$

I next want to show that we may modify $Q$ to obtain that the new $Q$ has $c = 0$. Indeed, modifying $Q$ means replacing $Q$ by $Q + u$ for some $u \in Cl_1^1(\mathfrak{g}) = \mathfrak{g}$. Now

$$(Q + u)^2 = Q^2 + [Q, u] + u^2$$

(since $Q$ and $u$ are odd). Now $u^2 = \frac{1}{2}(u, u)$ is a scalar. So if we set $Q' = Q + u$ and $c' = \mathrm{gr}_2(Q')^2$ we have, taking $\mathrm{gr}_2$ of the above equation,

$$c' = c + \{\Omega, u\} = c + du.$$

In other words, the cohomology class, $[c]$, is independent of the choice of the representative, $Q$. I claim that this cohomology class vanishes. More precisely, I claim that if we choose

$$Q = q(\Omega)$$

(where $q : \wedge\mathfrak{q} \to Cl(\mathfrak{q})$ is the quantization map), then $c = 0$. Indeed, let $X(dx)$ be the derivation of $Cl(\mathfrak{q})$ induced by the element of $o(\mathfrak{q})$ corresponding to $dx \in \wedge^2\mathfrak{q}$ and $Y(dx)$ the derivation of $\wedge(\mathfrak{q})$ corresponding to $dx$. The quantization map $q$ is defined purely in terms of the scalar product on $\mathfrak{g}$ and hence commutes with all orthogonal transformations, so

$$X(dx)Q = q(Y(dx)\Omega).$$

For any $y \in \mathfrak{g}$ we have

$$X(dx)y = \{dx, y\} = [q(dx), y],$$

and hence since $X(dx)$ and $[q(dx), \cdot]$ are derivations,

$$X(dx) = [q(dx), \cdot]$$

on all of $Cl(\mathfrak{g})$. Similarly $Y(dx) = \{dx, \cdot\}$ on all of $\wedge(\mathfrak{g})$. Now

$$[Q^2, x] = [Q, [Q, x]] = [Q, q(dx)]$$

since $dx = \mathrm{gr}_2[Q, ])$ and scalars commute with the entire Clifford algebra. So

$$[Q^2, x] = -X(dx)Q = -q(Y(dx)\Omega)$$

by the previous slide. So to show that for this choice of $Q$ we have $[Q^2, x] = 0$, we must show that

$$Y(dx)\Omega = 0.$$

For this, I will show that $Y(dx) = \operatorname{ad} x$ when acting on $\mathfrak{g}$. This is enough, for since $\Omega$ is defined purely in terms of the Lie bracket and the scalar product, it follows that $Y(dx)\Omega = 0$.

**Proof.**

For any $x, y, z \in \mathfrak{g}$ we have

$$(z, \{dx, y\}) = -(z, \{y, dx\}) = -(z, \{y, \{\Omega, x\}\}) = -(z, \{y, i(x)\Omega\})$$

$$= -(z, i(y)i(x)\Omega) = -i(z)i(y)i(x)\Omega = -(x, [y, z]_{\mathfrak{g}}) = ([x, y]_{\mathfrak{g}}, z).$$

$\square$

Here is a shorter proof that if we take $Q = q(\Omega)$ then $Q^2$ is a scalar: For any choice of $Q$ with $\mathrm{gr}_3\, Q = \Omega$, we know that $Q^2$ has filtration degree at most 2. But by the corollary to the corollary about quantization, we know that if we take $Q = q(\Omega)$ then $\sigma(Q^2)$ has no component of degree 2.

# The group $C^\times$.

Let $V$ be a vector space over the real numbers with a non-degenerate symmetric bilinear form which, in this section, I will denote by $(v, w)$ instead of $B(v, w)$. Also, I will denote the Clifford algebra of $V$ by $C$ since $V$ will be fixed throughout this section.

Let $C^\times$ denote the group of (multiplicatively) invertible elements of $C$. I will describe a subgroup $Pin(V)$ of $C^\times$ which double covers the orthogonal group $O(V)$ and a subgroup $Spin(V)$ of $Pin(V)$ which double covers $SO(V)$ where $SO(V)$ denotes the group of orthogonal transformations of $V$ with determinant 1.

I need a fact (due to Cartan and Dieudonné) about orthogonal groups.

# Reflections.

Let $v \in V$ be such that $(v, v) \neq 0$. The reflection in the hyperplane orthogonal to $v$ will be denoted by $\mathrm{r}_v$. So

$$\mathrm{r}_v(u) = u \text{ if } u \perp v, \quad \mathrm{r}_v(v) = -v$$

for any $u \in V$. We can combine these equations as

$$\mathrm{r}_v(u) = u - 2\frac{(u, v)}{(v, v)}v.$$

Direct computation using this formula shows that $\mathrm{r}_v$ is an isometry, i.e. that

$$(\mathrm{r}_v(u), \mathrm{r}_v(w)) = (u, w) \quad \forall\, u, w \in V.$$

Since $v^2 = \frac{1}{2}(v, v)1$ in the Clifford algebra, we see that $v$ is invertible with inverse

$$v^{-1} = \frac{2}{(v, v)} v.$$

So multiplying the defining identity

$$vu + uv = (u, v)1$$

by $v^{-1}$ gives

$$vuv^{-1} = -\mathrm{r}_v u. \tag{4}$$

# The Cartan-Dieudonné theorem.

**Theorem**

*If $V$ is a vector space over the reals with a non-degenerate scalar product, then every isometry of $V$ is a product of at most $n+1$ reflections where $n = \dim V$.*

**Proof by induction on $n$.** If $n = 1$ the only isometries are $u \mapsto \pm u$ and $\mathrm{r}_u(u) = -u$. The identity map is the square of the reflection.

Before proceeding with the induction, we prove

**Lemma**

*If $(u,u) = (w,w) \neq 0$ there is a reflection carrying $u$ to $\pm w$.*

**Proof of the lemma.** We have

$$(u+w, u+w) + (u-w, u-w) = 4(u,u) \neq 0.$$

So at least one of the summands on the left hand side is $\neq 0$. Replacing $w$ by $-w$ if necessary, we may assume that

$$v := u - w \quad \text{satisfies} \quad (v,v) \neq 0.$$

Then

$$(u, u-w) = \frac{1}{2}(u-w, u-w)$$

and so

$$\mathrm{r}_v(u) = u - 2\frac{(u, u-w)}{(u-w, u-w)}(u-w) = u - (u-w) = w. \qquad \square$$

# Proof of the Cartan- Dieudonné theorem.

Now to the induction. Assume we know the theorem for spaces of dimension $n-1$, and let $\tau$ be an isometry of the $n$-dimensional space $V$. Choose some $u \in V$ with $(u,u) \neq 0$, and set $w = \tau(u)$. By the lemma, there is a $v \in V$ with

$$\mathrm{r}_v u = \pm w$$

and hence

$$(\mathrm{r}_v)^{-1} \circ \tau u = \pm u$$

and so

$$\tau_1 := (\mathrm{r}_v)^{-1} \tau$$

is an isometry of $V$ which leaves the subspace $u^\perp$ invariant. The restriction of $\tau_1$ to $u^\perp$ can be written as a product of at most $n$ reflections in $u^\perp$, and each such reflection extends to a reflection in $V$ be setting it equal to the identity on the line through $u$. Thus we have written $\tau_1$ as the product of at most $n$ reflections, and hence $\tau$ is the product of at most $n+1$ reflections. $\square$

# The degree involution.

The map $u \mapsto -u$ is an isometry of $V$ and hence extends to an automorphism $\omega$ of $C$ which is an involution, i.e. $\omega^2 = 1$. Clearly $\omega = 1$ on $C_+$ and $\omega = -1$ on $C_-$. So $\omega$ is the degree derivation. The restriction of $\omega$ to any subspace invariant under $\omega$ has eigenvalues $\pm 1$. So if $U \subset C$ is invariant under $\omega$ then

$$U = U_0 \oplus U_1. \quad U_0 = U \cap C_0, \quad U_1 = U \cap C_1.$$

# Twisted conjugation and the Clifford group.

Recall that $C^\times$ denotes the group of invertible elements of $C$. Notice that if $a \in C^\times$ then

$$1 = \omega(aa^{-1}) = \omega(a)\omega(a^{-1})$$

so $C^\times$ is stable under $\omega$ and

$$\omega(a^{-1}) = \omega(a)^{-1}.$$

There is a representation of $C^\times$ on $C$ (called the twisted conjugation action) defined by

$$\mathrm{Twad}(a)u := \omega(a)ua^{-1}.$$

We have

$$\omega\left(\omega(a)\omega^{-1}(u)a^{-1}\right) = au\omega(a)^{-1}.$$

In other words,

$$\omega \circ \mathrm{Twad}(a) \circ \omega^{-1} = \mathrm{Twad}(\omega(a)).$$

# The Clifford group.

The **Clifford group** $\Gamma$ is defined to be the subgroup of $C^\times$ consisting of those elements which satisfy

$$\mathrm{Twad}(a)V = V.$$

Notice that if $\mathrm{Twad}(a)x \in V$ so is

$$-\omega(\mathrm{Twad}(a)x) = (\omega \circ \mathrm{Twad}(a)\omega^{-1})x = (\mathrm{Twad}(\omega(a))(x).$$

So $\Gamma$ is invariant under $\omega$.

Also, $\Gamma$ is invariant under the anti-automorphism $a \mapsto a^T$. Indeed, if $a \in \Gamma$ then $a^{-1} \in \Gamma$ and applying the canonical anti-automorphism to

$$\omega(a^{-1})xa \in V$$

gives

$$a^T x \omega(a^{-1})^T = a^T x \omega(a^T)^{-1} \in V$$

and applying $\omega$ gives

$$\omega(a^T)x(a^T)^{-1} \in V.$$

# Definition of $\overline{a}$.

$\Gamma$ is invariant under $\omega$. $\Gamma$ is invariant under $\omega$.

**Define** $\overline{a} := \omega(a^T)$.

Putting the above two facts together we get

$$a \in \Gamma \Rightarrow \overline{a} \in \Gamma.$$

For example, if $a = x_1 \cdots x_k,\ x_i \in V$ with no $(x_i, x_i) = 0$, then $a \in \Gamma$ and $a$ acts on $V$ as the product

$$r_{x_1} \cdots r_{x_k}$$

of the reflections $r_{x_i}$ and

$$a\overline{a} = (-1)^k (x_1, x_1) \cdots (x_k, x_k) 1,$$

a scalar multiple of 1.

In any event, we have, by definition, a representation $\Phi : \Gamma \to Gl(V)$. The purpose of the next few slides is to prove:

**Theorem**

*The following facts are true:*

- *The kernel of* $\Phi$ *consists of the non-zero multiples of* $1$.
- *The image of* $\Phi$ *is the orthogonal group* $O(V)$.
- $\Gamma$ *is generated by the* $x \in V$ *with* $(x, x) \neq 0$.
- *If* $a \in \Gamma$ *then* $a\bar{a}$ *is a scalar multiple of* $1$.

In the course of proving this theorem, we need to develop some facts about the center and anti-center of the Clifford algebra which are of independent interest.

# The element $\gamma$.

Let $e_1, \ldots, e_n$ be an "orthonormal" basis of $V$ and set

$$\gamma := e_1 e_2 \cdots e_n.$$

Notice that $\gamma$ is determined up to sign, that

$$\gamma^2 = (-1)^{n(n-1)/2+q} 1 \tag{5}$$

and

$$\gamma x = (-1)^{n-1} x \gamma, \quad \text{for} \quad x \in V$$

and hence

$$\gamma \cdot u = \omega(u)^{n-1} \gamma, \quad u \in C. \tag{6}$$

# The center.

**Definition**

The center $Z$ is defined as the set of all $z \in C$ such that

$$zu = uz \quad \forall\ u \in C.$$

Clearly $z \in Z$ if and only if

$$zx = xz \quad \forall x \in V.$$

$Z$ is invariant under $\omega$.

Indeed, if $z \in Z$ and $x \in V$ then

$$\omega(z)x = -\omega(z)\omega(x) = -\omega(zx) = -\omega(xz) = x\omega(z).$$

# The anti-center.

Similarly, the anti-center $AZ$ is defined as the set of all $a$ such that

$$au = \omega(u)a \quad \forall\ u \in C.$$

By a similar argument, it also is invariant under the degree involution.

So both $Z$ and $AZ$ are sums of their even and odd parts. Notice that

$$AZ_1 = \{0\}. \tag{7}$$

Indeed, if $a \in AZ_1$ we have $a\gamma = (-1)^n \gamma a$ since $a \in AZ$ and also $a\gamma = (-1)^{n-1}\gamma a$ since $a \in C_1$. So $a\gamma = 0$. But $\gamma^2 = \pm 1$ so $\gamma$ is invertible and so $a = 0$. $\square$

I also claim that

$$Z_0 = \mathbb{R}1. \tag{8}$$

# Proof by induction of the dimension, $n$, of $V$.

For $n = 1$, $C_0 = \mathbb{R}1$ and there is nothing to prove. So assume the result for spaces of dimension $n-1$ and let $y \in V$ be such that $(y, y) \neq 0$ and let $W = y^\perp$ (which is then also a space with a non-degenerate scalar product). Write $x \in V$ as $x = ry + w, \quad w \in y^\perp, \ r \in \mathbb{R}$ and $z \in Z_0$ as

$$z = b + yc, \quad b \in Cl(W)_0, \quad c \in Cl(W)_1.$$

So

$$\begin{aligned} zx &= rby - ry^2c + bw + ycw \\ xz &= ryb + ry^2c + wb + wyc \quad \text{so} \\ zx - xz &= -2ry^2c + bw - wb + 2y(cw + wc) \end{aligned}$$

since $yb = by$ and $yw = -wy$. So if $xz - zx = 0$ then $cw + wc = 0$ for all $w \in W$ and hence $c \in AZ(W)_1 = \{0\}$. Since $c = 0$, we conclude that $b \in Z(W)_0$ and hence is a scalar by induction. $\square$

I now claim that

- If $n$ is odd then $AZ = \{0\}$ and $Z = \mathbb{R}1 + \mathbb{R}\gamma$.
- If $n$ is even then $Z = \mathbb{R}1$ and $AZ = \mathbb{R}\gamma$.

**Proofs.** If $n$ is odd and $a \in AZ$ then $a\gamma = -\gamma a$ since $\gamma$ is odd. But $\gamma \in Z$ so $a = 0$. Multiplication by $\gamma$ maps $Z_0$ into $Z_1$ and is an isomorphism. By (8) we know that $Z_0 = \mathbb{R}1$ and multiplication by $\gamma$ sends 1 to $\gamma$.

If $n$ is even then $a \in Z$ implies $a\gamma = \gamma a$ but for general $a$ we have $\gamma a = \omega(a)\gamma$. So $\omega(a) = a$ and so $a \in Z_0$ so $a \in \mathbb{R}1$. Since multiplication by $\gamma$ (for $n$ even) interchanges $Z$ with $AZ$ we conclude that $AZ = \mathbb{R}\gamma$. $\square$

We can now prove the first assertion in Theorem 7 which we state here as a separate proposition:

If $b \in C$ satisfies

$$\omega(b)x = xb \quad \forall x \in V \tag{9}$$

then $b$ is a scalar (multiple of 1).

**Proof.** Break $b$ up into its even and odd parts:

$$b = b_0 + b_1.$$

Equation (9) says that

$$xb_0 = b_0 x \quad \text{and} \quad xb_1 = -b_1 x \quad \forall\, x \in V.$$

The second of these equations says that $b_1 \in AZ_1 = \{0\}$ so $b_1 = 0$ and the first of these equations says that $b_0 \in Z_0$ so $b_0$ is a scalar multiple of 1 by (8). $\square$

We now prove the last assertion in Theorem 7: that if $a \in \Gamma$ then $a\overline{a}$ is a scalar. Let $a \in \Gamma$ so that $\overline{a} \in \Gamma$ and for $x \in V$ let

$$y := \omega(\overline{a})x(\overline{a})^{-1}.$$

Since $y \in V$ we have $y^T = y$ and therefore

$$(\overline{a}^T)^{-1}x\omega(\overline{a})^T = \omega(\overline{a})x\overline{a}^{-1}$$

or

$$x\omega(\overline{a})^T\overline{a} = \overline{a}^T\omega(\overline{a})x$$

for all $x \in V$.
But $\omega(\overline{a})^T = a$ and $\overline{a}^T = \omega(a)$ so

$$xa\overline{a} = \omega(a\overline{a})x.$$

Now apply the fact we just proved. $\square$

# $a \mapsto \|a\|^2$ is a homomorphism from $\Gamma$ to the real numbers.

Notice that if $a, b \in \Gamma$ then

$$ab\overline{ab} = ab\overline{b} \cdot \overline{a} = (b\overline{b})a\overline{a}$$

since $b\overline{b}$ is a scalar multiple of 1. So if we define $\|a\|^2$ by

$$a\overline{a} = \|a\|^2 1$$

we see that $a \mapsto \|a\|^2$ is homomorphism from $\Gamma$ to the non-zero real numbers.

We have

$$\|\omega(a)\|^2 1 = \omega(a)\omega(\overline{a}) = \omega(a\overline{a}) = a\overline{a}$$

so

$$\|\omega(a)\|^2 = \|a\|^2.$$

Also,

$$\|\operatorname{Twad}(b)(a)\|^2 = \|\omega(b)ab^{-1}\|^2 = \|\omega(b)\|^2\|a\|^2\|b^{-1}\|^2 = \|a\|^2.$$

For $x \in V$ we have $\overline{x} = -x$ so

$$\|x\|^2 = -\frac{1}{2}(x, x).$$

Hence

$$(\mathrm{Twad}(a)(x), \mathrm{Twad}(a)(x)) = -2\|\mathrm{Twad}(a)(x)\|^2$$

$$= -2\|x\|^2 = (x, x).$$

In other words

The restriction of $\mathrm{Twad}(a)$ to $V$ is an isometry, and therefore this restriction defines a homomorphism, $\Phi$ from $\Gamma$ to $O(V)$.

To complete the proof of the second assertion in Theorem 7 we need to show that

**The homomorphism $\Phi$ is surjective**.

**Proof:** We have seen that every reflection through a hyperplane orthogonal to an $x$ with $(x, x) \neq 0$ is in the image of this homomorphism and that every element of $O(V)$ is a product of at most $n + 1$ such reflections. $\square$

Finally, we need to prove the third assertion in Theorem 7:

The elements $x \in V$ with $(x, x) \neq 0$ generate $\Gamma$.

**Proof.** Let $\tau = \Phi(a)$ and write $\tau$ as the product of some number of reflections so that $\tau = \Phi(x_1, \cdots x_k)$. Thus $a^{-1}x_1 \cdots x_k$ acts as the identity on $V$ so $a^{-1}x_1 \cdots x_k = \lambda \cdot 1$ for some scalar $\lambda$. So

$$a = \lambda^{-1}x_1 \ldots x_k = (\lambda^{-1}x_1)x_2 \cdots x_k. \quad \square$$

Notice that since $\Phi(a) \in O(V)$ we have $\det \Phi(a) = \pm 1$, and since the sign is $-1$ for $x \in V$ it follows that

$$\det \Phi(a)a = \omega(a). \tag{10}$$

# Repeat of the statement of Theorem 7.

The **Clifford group** $\Gamma$ is defined to be the subgroup of $C^\times$ consisting of those elements which satisfy

$$\mathrm{Twad}(a)V = V.$$

So we have a representation $\Phi : \Gamma \to Gl(V)$.

**Theorem**

- The kernel of $\Phi$ consists of the non-zero multiples of 1.
- The image of $\Phi$ is the orthogonal group $O(V)$.
- $\Gamma$ is generated by the $x \in V$ with $(x, x) \neq 0$.
- If $a \in \Gamma$ then $a\overline{a}$ is a scalar multiple of 1.

# The group $Pin(V)$.

This is defined as the subgroup of $\Gamma$ consisting of those $a$ with $\|a\|^2 = \pm 1$. If $b \in \Gamma$ and we define

$$a := \left| \|b\|^2 \right|^{-1} b$$

then $\Phi(a) = \Phi(b)$ and $\|a\|^2 = \pm 1$. So the restriction of $\Phi$ to $Pin(V)$ is still surjective, and $Pin(V)$ is generated by the elements $x \in V$ with $(x, x) = \pm 1$.

The kernel of the restriction of $\Phi$ to $Pin(V)$ is a scalar which must be $\pm 1$. So we get an exact sequence

$$1 \to S^0 \to Pin(V) \to O(V) \to 1$$

of groups where $S^0 = \{\pm 1\}$, where the second arrow is inclusion, and where third arrow is the restriction of $\Phi$ to $Pin(V)$.

# The group $Spin(V)$.

This is the subgroup of $Pin(V)$ consisting of the even elements, i.e. those satisfying

$$\omega(a) = a.$$

Since

$$\det \Phi(a)a = \omega(a),$$

the image of $Spin(V)$ under $\Phi$ is $SO(V)$, the subgroup of $O(V)$ consisting of elements of determinant one. The kernel of restriction of $\Phi$ to $Spin(V)$ is again $\pm 1$ since these elements are even. So we get the exact sequence of groups,

$$1 \to S^0 \to Spin(V) \to SO(V) \to 1.$$

# The positive definite case.

If $V$ is positive (or negative) definite and $n \geq 2$, we can find elements $x$ and $y$ in $V$ which are orthogonal and such that $(x, x) = 1 = (y, y)$ (in the positive definite case) or $(x, x) = (y, y) = -1$ in the negative definite case. In either case we have

$$(xy)^2 = -1.$$

So exponentiating in the Clifford algebra gives (for $t \in \mathbb{R}$)

$$exp(t(xy)) = 1 + txy + \frac{1}{2}t^2(xy)^2 + \cdots = \cos t + (\sin t)xy.$$

Setting $t = \pi$ we see that $-1$ belongs to the connected component of the identity in $Spin(V)$.

Also (in the positive definite case, say)

$$(\cos t + (\sin t)xy)^{-1} = \cos t - (\sin t)xy$$

and so

$$\begin{aligned}(\cos t + (\sin t)xy)x(\cos t + (\sin t)xy)^{-1} \\ = ((\cos t)x - (\sin t)y)(\cos t - (\sin t)xy) \\ = (\cos 2t)x - (\sin 2t)y.\end{aligned}$$

In other words, $\Phi(\exp txy))$ is rotation through angle $-2t$ in the $x, y$ plane. This clearly shows the nature of the double covering. For $n \geq 3$ we know that the fundamental group of the orthogonal group (in the definite case) is $\mathbb{Z}_2$ so $Spin(V)$ is the universal cover.

In the non-definite case, if either $p \geq 2$ or $q \geq 2$ we can apply the above result to a definite plane and still conclude that $-1$ is in the connected component of $Spin(V)$.

# Minkowski space

Let $e_0, e_1, e_2, e_3$ be an "orthonormal" basis of a four dimensional space. For the case $p = 1, q = 3$ we have $e_0^2 = 1$ and $\Phi(e_0)$ is reflection in the plane orthogonal to $e_0$, i.e. the "time reversal" operator $T$. Also,

$$(e_1 e_2 e_3)^2 = 1$$

and $\Phi(e_1 e_2 e_3)$ is the product of three reflections in orthogonal planes in the space spanned by $e_1, e_2, e_3$ and so is the "parity transformation" $P$

$$Pe_0 = e_0, \quad Pe_i = -e_i, \quad i = 1, 2, 3.$$

For the case $p = 3, q = 1$ we have $e_0^2 = -1$ and $(e_1 e_2 e_3)^2 = -1$. so in this case the elements in $Pin(V)$ which cover $T$ and $P$ are of order 4, not of order 2.

We now restrict ourselves to the case where the symmetric quadratic form is split. Over the reals, this means that $B$ has signature $(m, m)$ where the dimension of $V$ is $n = 2m$. Over $\mathbb{C}$ this means that $V$ is even dimensional.

We will examine this case in detail in the next lecture.