Lipschitz Normal Embeddings in the Space of Matrices

Maria Aparecida Soares Ruas - ICMC-USP

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Introduction

Lipschitz normal embeddings in the space of matrices, Dmitry Kerner, Helge Moeller Pedersen, R., arXiv:1703.04520.

- *Bilipschitz geometry of generic determinantal varieties*

Recall that a map $f : X \to Y$, X and Y metric spaces is bilipschitz if f is a Lipschitz homeomorphism whose inverse is also Lipschitz.

• $(X, 0)$ germ of algebraic (analytic) variety over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, embedded in $(\mathbb{K}^N, 0)$.

We can define two natural metrics on X.

• The outer metric

$$d_{out}(x, y) := ||x - y||_{\mathbb{K}^N}$$

i.e. the restriction of the Euclidean metric to $(X, 0)$.

• The inner metric

$$d_{in}(x, y) := \inf_{\gamma} \left\{ length_{\mathbb{K}^N}(\gamma) \mid \gamma : [0, 1] \to X, \gamma(0) = x,\ \gamma(1) = y \right\}. \quad (1)$$

Both metrics are independent of the choice of the embedding up to bilipschitz equivalence.

It is clear that

$$d_{out}(x,y) \leq d_{in}(x,y).$$

The other direction is in general not true, and one says that

Definition

$(X,0)$ is Lipschitz normally embedded (L.N.E.) if the outer and inner metrics are equivalent.

That is, if there exists a constant $K > 0$ such that

$$\frac{d_{in}(x,y)}{K} \leq d_{out}(x,y).$$

Example

The plane curve $x^3 - y^2 = 0$ *is not L.N.E.. In fact,*

$d_{out}(t^2, t^3), (t^2, -t^3)) = 2|t|^3$ *and* $d_{in}((t^2, t^3), (t^2, -t^3)) = 2|t|^2 + o(t^2)$.

This implies that

$$\frac{d_{in}((t^2, t^3), (t^2, -t^3))}{d_{out}((t^2, t^3), (t^2, -t^3))}$$

is unbounded as $t \to 0$, *hence there cannot exist a* K *as in the previous definition.*

An irreducible complex plane curve is L.N.E. if and only if it is smooth.

Main result

Let $Mat_{m\times n}(\mathbb{K})$ be the space of $m \times n$ matrices over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Let $X \subset Mat_{m\times n}(\mathbb{K})$.

$X_r := \{A \in X \,|\mathrm{rank}\, A = r\}$, $\overline{X}_r$ its topological closure

(Generic determinantal varieties)

When $X = Mat_{m\times n}(\mathbb{C})$, $\overline{X}_r$ is an irreducible algebraic variety such that:

- $\mathrm{cod}\, \overline{X}_r = (m-r)(n-r)$
- $Sing(\overline{X}_r) = \overline{X}_{r-1}$
- The decomposition $\overline{X}_r = \bigcup_{s\le r} X_s$ is a Whitney stratification.

Theorem

(Kerner, Pedersen, R.) Let X be one of the following sets:

(1) $Mat_{m\times n}(\mathbb{K})$

(2) $n \times n$ symmetric matrices over $\mathbb{K}$

(3) $n \times n$ antisymmetric matrices over $\mathbb{K}$.

Then $\overline{X}_r$ and X_r, $r \leq m \leq n$ are Lipschitz Normally Embedded.

Properties of the varieties X_r and $\overline{X}_r$

- X_r and $\overline{X}_r$ are G-invariant as follows:
  $X = Mat_{m \times n}(\mathbb{C}),\ G = U(m) \times U(n)$ and
  $X = Mat_{m \times n}(\mathbb{R}),\ G = O(m) \times O(n)$.
- $\overline{X}_r$ is a (metric) cone.
- The local structure of $\overline{X}_r$ and the *controlled path-connectedness* of the connected components of X_r.

L.N.E. of $\overline{X}_r$.

Theorem

Let $\mathbb{K} \in \mathbb{R}, \mathbb{C}$ *and* X *one of the spaces* $Mat_{m\times n}(\mathbb{K})$, $Mat^{sym}_{n\times n}(\mathbb{K})$, $Mat^{skew-sym}_{n\times n}(\mathbb{K})$. *For any* $1 \leq r \leq m \leq n$ *and* $A, B \in \overline{X_r}$ *holds:*

$$\frac{d_{in}^{\overline{X_r}}(A,B)}{2\sqrt{2}} \leq d_{out}(A,B) \leq d_{in}^{\overline{X_r}}(A,B)$$

.

Proof.

We let $X = Mat_{m\times n}(\mathbb{K})$
$U(m) \times U(n)$ acts on $Mat_{m\times n}(\mathbb{K})$ by $A \to UAV$.
As rank $A \leq r$, we can reduce A to the form.

$$\begin{bmatrix} A_1 & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}, \quad A_1 \in Mat_{r\times r}(\mathbb{K}).$$

Present B accordingly: $\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$. Then:

$$d_{out}(A,B) = \sqrt{||A_1 - B_1||^2 + ||B_2||^2 + ||B_3||^2 + ||B_4||^2}$$

□

Consider the path $B(t) = \begin{bmatrix} B_1 & tB_2 \\ tB_3 & t^2 B_4 \end{bmatrix}$ for $t \in [0, 1]$. Scaling a particular row/column do not increase the rank, then $B(t) \in \overline{X_r}$ for any $t \in [0, 1]$.

Therefore we get an algebraic curve (inside $\overline{X_r}$) that connects $B = B(1)$ to $B(0) = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$.

The length of this path is: $\int_0^1 \sqrt{||B_2||^2 + ||B_3||^2 + 4t^2||B_4||^2} dt$.

It remains to move from $B(0)$ to A.
In total we get:

$$d_{in}^{\overline{X_r}}(A, B) \leq \int_0^1 \sqrt{||B_2||^2 + ||B_3||^2 + 4t^2||B_4||^2} dt + ||A_1 - B_1||.$$

Now we use the bounds

$$\int_0^1 \sqrt{||B_2||^2 + ||B_3||^2 + 4t^2||B_4||^2} dt < 2\sqrt{||B_2||^2 + ||B_3||^2 + ||B_4||^2}$$

and $x + y \leq \sqrt{2(x^2 + y^2)}$ to get:

$$\begin{aligned} d_{in}^{\overline{X}_r}(A, B) &< 2\sqrt{||B_2||^2 + ||B_3||^2 + ||B_4||^2} + ||A_1 - B_1|| \leq \\ &\leq 2\sqrt{2}\sqrt{||A_1 - B_1||^2 + ||B_2||^2 + ||B_3||^2 + ||B_4||^2} \\ &= 2\sqrt{2} \cdot d_{out}(A, B). \end{aligned}$$

L.N.E. of X_r

Theorem

Let $\mathbb{K} \in \mathbb{R}, \mathbb{C}$ and X be one of the spaces $Mat_{m\times n}(\mathbb{K})$, $Mat^{sym}_{n\times n}(\mathbb{K})$, $Mat^{skew-sym}_{n\times n}(\mathbb{K})$. Suppose A, B belong to the same connected component of X_r, for some $r \leq m$. Then

$$\frac{d_{in}^{X_r}(A,B)}{2\sqrt{2}} \leq d_{out}(A,B) \leq d_{in}^{X_r}(A,B).$$

Notice that when $\mathbb{K} = \mathbb{C}$, the varieties X_r are connected, however this does not hold for real matrices. We give the general lines of the proof in the case $\mathbb{K} = \mathbb{C}$.

Proof.

Step 1. (Reduction to the case of X_n.) As in the proof for $\overline{X_r}$, A can be reduced to the form $\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$. And B is brought to $\begin{bmatrix} B_1 & * \\ * & * \end{bmatrix}$.

It might happen that $rank(B_1) < r$. To avoid this we can take arbitrarily small but generic deformation of B inside X_r.

Now, as $rank(B_1) = r$, we can take the path $B(t) = \begin{bmatrix} B_1 & t* \\ t* & t^2* \end{bmatrix}$, and as before, the length of this path is less than $2 \cdot \sqrt{(\ldots)}$.

It remains to connect the matrices $\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$ inside X_r by a path of the total length $\leq 2d_{out}(A_1, B_1) + \epsilon$.

So, the initial question has been reduced to the stratum X_n of square matrices.

□

Step 2.
Consider the straight segment $[A, B] \subset X$, $A, B \in X_n$.

By algebraicity of the strata, it intersects $\overline{X_{n-1}}$ in a finite number of points which is at most $\deg(\overline{X_{n-1}})$.

Now, by the controlled path connectedness, we can deform the path slightly at each of these point to push it into the stratum X_n.

Hence we get a path inside X_n of length $\leq d_{out}(A, B) + \epsilon$. Together with the path $B(t)$ of step 1 this finishes the proof.

Example.

Let $V \subset Mat_{3\times 3}(\mathbb{C})$ be the linear subspace given as the image of the following map $F : \mathbb{C}^3 \to Mat_{3\times 3}(\mathbb{C})$:

$$F(x, y, z) = \begin{pmatrix} x & 0 & z \\ y & x & 0 \\ 0 & y & x \end{pmatrix}.$$

Let $Y := V \cap \overline{X}_2$, where $\overline{X}_2$ is the set of matrices in $Mat_{3\times 3}(\mathbb{C})$ with zero determinant, which is Lipschitz normally embedded.

The variety $Y = V(x^3 - y^2 z)$ is a family of cusps degenerating to a line.

Y being Lipschitz normally embedded would imply that the cusp $x^3 - y^2 = 0$ is Lipschitz normally embedded, a contradiction.

Proposition

Let $V \subset X = Mat_{m\times n}$ *be a linear subspace intersecting* X_r *transversely for all* $s \neq 0,\ s \leq r$ *Then* $Y := V \cap \overline{X}_r$ *is Lipschitz normally embedded.*

Merci Beaucoup !

Muchas gracias !

Thank you very much !

Muito obrigada !