Notes on Differential Geometry

Fernando Manfio
University of São Paulo

2024

Contents

Chapter 1

Regular surfaces

In these notes the term differentiable means differentiable of class C^∞.

1.1 Regular surfaces

In few words, a regular surface in $\mathbb{R}^3$ is a subset that is locally homeomorphic to an open subset of $\mathbb{R}^2$.

Definition 1.1.1. A subset $M \subset \mathbb{R}^3$ is called a *regular surface* if, for each point $p \in M$, there exists an open set $V \subset \mathbb{R}^3$, with $p \in V$, and a homeomorphism $\varphi : U \to M \cap V$ defined on an open set $U \subset \mathbb{R}^2$, such that

(a) φ is differentiable,

(b) For each point $x \in U$, the differential $\mathrm{d}\varphi(x) : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

The mapping φ is called a *parametrization* of M around p, and the subset $M \cap V$ is called a *coordinate neighborhood* of M. This means that M is endowed with the induced topology of $\mathbb{R}^3$, and therefore any regular surface is, in particular, a topological subspace of $\mathbb{R}^3$.

The condition that φ is differentiable means that if we write

$$\varphi(x_1, x_2) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2), \varphi_3(x_1, x_2)),$$

then the coordinate functions $\varphi_1, \varphi_2, \varphi_3$ have continuous partial derivatives of all orders in the open set U.

The condition that $\mathrm{d}\varphi(x) : \mathbb{R}^2 \to \mathbb{R}^3$ is a injective linear map is equivalent to any of the following conditions:

(a) The set $\{\mathrm{d}\varphi(x) \cdot e_i : 1 \leq i \leq 2\}$ is linearly independent, where $\{e_1, e_2\}$ denotes the canonical basis of $\mathbb{R}^2$.

(b) The Jacobian matrix $\mathrm{d}\varphi(x)$ has rank two at any point $x \in U$.

Example 1.1.2. Any two-dimensional vector space $E \subset \mathbb{R}^3$ is a regular surface. In fact, consider a linear isomorphism $T\colon E \to \mathbb{R}^2$, and endow E with the unique topology (induced of $\mathbb{R}^3$) that makes T a homeomorphism. Since any linear mapping into $\mathbb{R}^2$ is differentiable, it follows that T is a diffeomorphism, and thus T is a global parametrization of E.

Example 1.1.3. Let us show that the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is a regular surface. Fix a point $p \in \mathbb{S}^2$ other than the north pole $N = (0, 0, 1)$ and consider the stereographic projection $\pi_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$. We already know that π_N is a homeomorphism, whose inverse is the map $\varphi : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{N\}$ given by

$$\varphi(x) = \left(\frac{2x_1}{\|x\|^2 + 1}, \frac{2x_2}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1} \right),$$

for every point $x = (x_1, x_2) \in \mathbb{R}^2$. Since each coordinate function of φ is differentiable, it follows that φ is also differentiable. It is straightforward to check that $d\varphi(x)$ has rank two at any point $x \in \mathbb{R}^2$. Finally, if $p = N$, just consider the stereographic projection π_S relative to the south pole $S \in \mathbb{S}^2$.

Example 1.1.4. The graph of a differentiable function $f : U \to \mathbb{R}$, defined in an open set $U \subset \mathbb{R}^2$, is a regular surface. In fact, denoting by $\mathrm{Gr}(f)$ the graph of f, let us show that the map $\varphi : U \to \mathbb{R}^3$ given by

$$\varphi(x) = (x, f(x)),$$

is a global parametrization of $\mathrm{Gr}(f)$. Since f is differentiable, the same holds for φ. Each point $(x, f(x)) \in \mathrm{Gr}(f)$ is the image under φ of the unique point $x \in U$, and φ is therefore injective. Moreover, the restriction to $\mathrm{Gr}(f)$ of the projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$ is a inverse to φ, and this shows that φ^{-1} is also continuous. It follows that φ is a homeomorphism. Finally, it is easy to see that $\mathrm{d}\varphi(x)$ has rank two at any point $x \in U$.

The following result provides a local converse of Example 1.1.4. More precisely, any regular surface is locally the graph of a differentiable function.

Proposition 1.1.5. Given a regular surface $M \subset \mathbb{R}^3$ and a point $p \in M$, there exist an open set $U \subset \mathbb{R}^2$, an open set $V \subset \mathbb{R}^3$ with $p \in V$, and a differentiable function $g : U \to \mathbb{R}$ such that $M \cap V = \mathrm{Gr}(g)$.

Proof. Fix a point $p \in M$ and consider a parametrization $\varphi : U \to \varphi(U)$ of M, with $p = \varphi(x)$. Since $E = \mathrm{d}\varphi(x)(\mathbb{R}^2)$ is a two-dimensional vetor subspace of $\mathbb{R}^3$, there exists an orthogonal decomposition $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ such that the projection $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ maps E isomorphically onto $\mathbb{R}^2$, and define the map

$$\eta = \pi \circ \varphi : U \to \mathbb{R}^2.$$

Since $\mathrm{d}\eta(q) = \pi \circ \mathrm{d}\varphi(q)$ is a linear isomorphism, it follows from the inverse function theorem that there exist a open set $W \subset \mathbb{R}^2$, with $q \in W \subset U$, such that $\eta|_W : W \to \eta(W) = Z$ is a diffeomorphism. Define

$$\xi = (\eta|_W)^{-1} : Z \to W \quad \text{and} \quad \psi = \varphi \circ \xi.$$

It follows that ψ is also a parametrization of M and

$$\pi \circ \psi = \pi \circ (\varphi \circ \xi) = \eta \circ \xi = id.$$

According to the orthogonal decomposition $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$, it follows from the above equality that the first coordinate of $\psi(x)$ is x. Let us denote by $g(x)$ the second one. Thus,

$$\psi(Z) = \varphi(W) = \{(x, g(x)) : x \in W\}$$

for some differentiable function $g : W \to \mathbb{R}$. Since φ is a open map, one has

$$\varphi(W) = M \cap V = \mathrm{Gr}(g),$$

for some open set $V \subset \mathbb{R}^3$, with $p \in V$. $\square$

Let us look a simple application of Proposition 1.1.5.

Example 1.1.6. Let us consider the *one-sheeted cone* $M \subset \mathbb{R}^3$ given by

$$M = \{(x, y, z) : x^2 + y^2 = z^2, z \geq 0\}.$$

We will show that M is not a regular surface. If M were a regular surface then, by virtue of Proposition 1.1.5, M would be locally a graph of a differential function around $(0, 0, 0)$. More precisely, there exist open sets $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^3$, with $0 \in V$, and a differentiable function $g : U \to \mathbb{R}$ such that $M \cap V = \mathrm{Gr}(g)$. Note that, according to a decomposition $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$, the only possibility for $M \cap V$ to be a graph is for the second factor to be the axis-z. Thus, it follows that $g = f|_U$, where $f(x, y) = \sqrt{x^2 + y^2}$. However, f is not differentiable at $(0, 0)$.

Let $f : V \to \mathbb{R}$ be a differentiable function, defined in an open set $V \subset \mathbb{R}^3$. We say that a point $p \in V$ is a *regular point* of f if the direrential $\mathrm{d}f(p)$ is surjective, that is, $\mathrm{d}f(p) \neq 0$. A point $c \in \mathbb{R}$ is called a *regular value* of f if the inverse image $f^{-1}(c)$ contains only regular points of f. Notice that any point $c \notin f(V)$ is trivially a regular value of f.

Proposition 1.1.7. Let $f : V \to \mathbb{R}$ be a differentiable function, defined in an open set $V \subset \mathbb{R}^3$, and $c \in \mathbb{R}$ be a regular value of f. If $f^{-1}(c) \neq \emptyset$, then $M = f^{-1}(c)$ is a regular surface.

Proof. By virtue of Example 1.1.4, it suffices to prove that M is locally graph of some differentiable function. Given a point $p \in M$, with $p = (x_0, y_0, z_0)$, we can assume that $\frac{\partial f}{\partial z}(p) \neq 0$. Therefore, it follows from the implicit function theorem that there exist an open set $W = U \times I$, where U is an open set of $\mathbb{R}^2$ with $(x_0, y_0) \in U$, and I is an open interval with $z_0 \in I$, and a differentiable function $g : U \to \mathbb{R}$ such that

$$f((x, y), g(x, y)) = c,$$

for every $(x, y) \in U$. This proves that $M \cap W = \mathrm{Gr}(g)$. $\square$

Example 1.1.8. The unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ can be described as the inverse image $f^{-1}(1)$ of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x) = \|x\|^2 = \langle x, x \rangle,$$

for every $x \in \mathbb{R}^3$. Notice that f is differentiable and, for any point $p \in \mathbb{R}^3$ and any vector $v \in \mathbb{R}^3$, we obtain

$$\mathrm{d}f(p) \cdot v = 2\langle p, v \rangle.$$

This implies that $0 \in \mathbb{R}^3$ is the unique critical point of f. Since $f(0) = 0 \neq 1$, we conclude that 1 is regular value of f. Therefore, the sphere $\mathbb{S}^2$ is a regular surface as we have already seen.

Remark 1.1.9. The inverse image $f^{-1}(c)$ can be a regular surface without c being a regular value of f. For instance, consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = z^2$. Note that f is differentiable and $f^{-1}(0)$ is the plane-xy, which is a regular surface in $\mathbb{R}^3$. However, the point $0 \in \mathbb{R}$ is not a regular value of f, because $\mathrm{d}f(x, y, 0) = 0$, for every $(x, y, 0) \in f^{-1}(0)$.

The following result is a converse of Proposition 1.1.7.

Theorem 1.1.10. *Any regular surface M in $\mathbb{R}^3$ is locally the inverse image of a regular value. More precisely, given a point $p \in M$, there exist an open set $V \subset \mathbb{R}^3$ with $p \in V$, and a differentiable function $f\colon V \to \mathbb{R}$ such that $M \cap V = f^{-1}(0)$, where $0 \in \mathbb{R}$ is a regular value of f.*

Proof. It follows from Proposition 1.1.5 that there exists an open set $V \subset \mathbb{R}^3$, with $p \in V$, such that $M \cap V = \mathrm{Gr}(g)$, where $g\colon U \to \mathbb{R}$ is a differentiable function defined in an open set $U \subset \mathbb{R}^2$. Define a function $f\colon V \to \mathbb{R}$ by $f(x, y) = y - g(x)$. By construction, one has

$$M \cap V = \mathrm{Gr}(g) = f^{-1}(0).$$

It suffices to prove that $\mathrm{d}f(x, y)$ is surjective at any point $(x, y) \in f^{-1}(0)$. In fact, given $(x, y) \in f^{-1}(0)$ and $(u, v) \in \mathbb{R}^3$, we obtain:

$$\begin{aligned} \mathrm{d}f(x,y)\cdot(u,v) &= \mathrm{d}f(x,y)\cdot(u,0) + \mathrm{d}f(x,y)\cdot(0,v) \\ &= \mathrm{Id}(0) - \mathrm{d}g(x)\cdot u + \mathrm{Id}(v) - \mathrm{d}g(x)\cdot 0 \\ &= v - \mathrm{d}g(x)\cdot u. \end{aligned}$$

Therefore, given $v \in \mathbb{R}$, one has $\mathrm{d}f(x, y) \cdot (0, v) = v$, and this proves that 0 is a regular value of f. $\square$

1.2 Differentiable mappings between surfaces

In this section we will define what it means for a map $f\colon M \to N$, between two regular surfaces M and N, to be differentiable at a point $p \in M$.

Definition 1.2.1. A map $f : M \to N$, between the regular surfaces M and N, is said to be *differentiable* at a point $p \in M$ if there exist parametrizations $\varphi : U \to \varphi(U)$ of M and $\psi : V \to \psi(V)$ of N, with $f(\varphi(U)) \subset \psi(V)$ and $p = \varphi(x)$, such that the map

$$\psi^{-1} \circ f \circ \varphi : U \to V \tag{1.1}$$

is differentiable at $x \in U$.

The map given in (1.1) is called a *representation* of f in terms of the parametrizations φ and ψ. We have to show that this definition does not depend on the choice of parametrizations. In fact, consider parametrizations $\varphi'\colon U' \to \varphi'(U')$ of M and $\psi'\colon V' \to \psi'(V')$ of N, with $p \in \varphi'(U')$ and $f(\varphi'(U')) \subset \psi'(V')$. In the intersection $\varphi'^{-1}(\varphi(U) \cap \varphi'(U'))$, one has

$$\psi'^{-1} \circ f \circ \varphi' = (\psi'^{-1} \circ \psi) \circ (\psi^{-1} \circ f \circ \varphi) \circ (\varphi^{-1} \circ \varphi').$$

Therefore, the differentiability of f will be well-defined if we prove that $\psi'^{-1} \circ \psi$ and $\varphi^{-1} \circ \varphi'$ are differentiable. But this will be a consequence of the following lemma.

Let $\varphi : U \to \varphi(U)$ and $\psi : V \to \psi(V)$ be two parametrizations of a regular surface M such that $\varphi(U) \cap \psi(V) \neq \emptyset$. The map

$$\psi^{-1} \circ \varphi : \varphi^{-1}(W) \to \psi^{-1}(W), \tag{1.2}$$

is called the *change of coordinates* between the parametrizations φ and ψ, where $W = \varphi(U) \cap \psi(V)$.

Lemma 1.2.2. The change of coordinates (1.2) is a diffeomorphism.

Proof. Fix a point $p \in \varphi(U) \cap \psi(V)$, with $p = \varphi(q)$. Since $d\varphi(q)$ is injective, and writing

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

we can assume without loss of generality that $\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$. Define a map $\xi : U \times \mathbb{R} \to \mathbb{R}^3$ by

$$\xi(u, v, w) = (x(u, v), y(u, v), z(u, v) + w).$$

ξ is clearly differentiable and $\xi|_{U \times \{0\}} = \varphi$, thus

$$\det(\mathrm{d}\xi(q, 0)) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & 1 \end{pmatrix} = \frac{\partial(x, y)}{\partial(u, v)}(q) \neq 0.$$

Therefore, it follows from inverse function theorem that there exists an open set $K \subset \mathbb{R}^3$, with $(q, 0) \in K$, such that $\xi|_K : K \to \xi(K)$ is a diffeomorphism. Note that $Z = \xi(K)$ is an open set of $\mathbb{R}^3$, with $p \in Z$. Since $\xi|_{U \times \{0\}} = \varphi$, one has

$$\xi^{-1}|_{\varphi(U)\cap Z} = \varphi^{-1}|_{\varphi(U)\cap Z}.$$

On the other hand, since $\varphi(U) \cap Z$ is an open set of M and ψ is a homeomorphism, it follow that $\psi^{-1}(\varphi(U) \cap Z)$ is an open set of V. Therefore,

$$\varphi^{-1} \circ \psi|_{\psi^{-1}(\varphi(U)\cap Z)} = \xi^{-1} \circ \psi|_{\psi^{-1}(\varphi(U)\cap Z)}$$

is differentiable as a composition of differentiable maps. Analogously, we can show that $\psi^{-1} \circ \varphi$ is differentiable, and thus it is a diffeomorphism. □

Let us explore some consequences.

Corollary 1.2.3. Let M, N be regular surfaces, and assume that $M \subset V$, where V is an open set of $\mathbb{R}^3$, and that $f\colon V \to \mathbb{R}^3$ is a differentiable map such that $f(M) \subset N$. Then the restriction $f|_M\colon M \to N$ is a differentiable map.

Proof. Given a point $p \in M$, consider parametrizations $\varphi\colon U \to \varphi(U)$ of M an $\psi\colon V \to \psi(V)$ of N, with $p \in \varphi(U)$ and $f(\varphi(U)) \subset \psi(V)$. Then, the map

$$\psi^{-1} \circ f \circ \varphi\colon U \to V$$

is differentiable. □

Example 1.2.4. The map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (ax, by, cz)$, where a, b, c are positive real numbers, is clearly differentiable. The restriction of f to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is differentiable. In fact, $f|_{\mathbb{S}^2}$ is a differentiable map of the sphere $\mathbb{S}^2$ into the ellipsoid $\mathcal{E}$.

Remark 1.2.5. In the case of a map $f\colon M \to \mathbb{R}^2$, of a regular surface M into $\mathbb{R}^2$, the Definition 1.2.1 takes a rather simpler form. Namely, in this case, f is differentiable at $p \in M$ if there exists a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$, such that

$$f \circ \varphi\colon U \to \mathbb{R}^2$$

is differentiable at $x \in U$. In fact, just consider ψ equal to the identity in Definition 1.2.1.

Corollary 1.2.6. If $\varphi\colon U \to \varphi(U)$ is a parametrization of a regular surface M, then $\varphi^{-1}\colon \varphi(U) \to \mathbb{R}^2$ is also differentiable.

Proof. Given a point $p \in \varphi(U)$, consider the parametrization $\varphi\colon U \to \varphi(U)$ of M. One has $p \in \varphi(U)$ and the representation of φ^{-1} in terms of φ is just the identity map, which is differentiable. □

Two regular surfaces M and N are *diffeomorphic* if there exists a bijective differentiable map $f\colon M \to N$, whose inverse $f^{-1}\colon N \to M$ is also differentiable. In this case, f is called a *diffeomorphism* from M to N. In particular, it follows from Corollary 1.2.6 that, if $\varphi\colon U \to \varphi(U)$ is a parametrization of a regular surface M, then U and $\varphi(U)$ are diffeomorphic.

Finally, we will now give a definition for a differentiable function on a regular surface.

Definition 1.2.7. A function $f\colon M \to \mathbb{R}$, defined on a regular surface M, is said to be *differentiable* at a point $p \in M$ if there exists a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$, such that the composition

$$f \circ \varphi\colon U \to \mathbb{R}$$

is differentiable at $x \in U$.

It follows from Lemma 1.2.2 that the Definition 1.2.7 does not depend on the choice of the parametrization φ. In fact, if $\psi\colon V \to \psi(V)$ is another parametrization of M, with $p = \psi(y)$, then

$$f \circ \psi = (f \circ \varphi) \circ (\varphi^{-1} \circ \psi)$$

is also differentiable.

Corollary 1.2.8. If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, then the restriction of f to any regular surface M is a differentiable function on M.

Proof. For any point $p \in M$ and any parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$, the function $f \circ \varphi\colon U \to \mathbb{R}$ is differentiable at $x \in U$. $\square$

Example 1.2.9. Given a regular surface M and a unit vector $v \in \mathbb{R}^3$, consider the height function $f\colon M \to \mathbb{R}$ relative to v, given by $f(p) = \langle p, v \rangle$ for every $p \in M$. It follows immediatly from Corollary 1.2.8 that f is differentiable.

1.3 The tangent plane

In this section we will define the tangent plane to a regular surface M at a point $p \in M$. Before stating the concepts we will need some terminology.

A *parametrized differentiable curve* in $\mathbb{R}^3$ is just a differentiable map $\alpha\colon I \to \mathbb{R}^3$ defined in an open interval $I \subset \mathbb{R}$. The term differentiable means that α is a correspondence which maps each instant $t \in I$ into a point $\alpha(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$ in such way that the functions $x(t), y(t), z(t)$ are differentiable. The variable t is called the *parameter* of the curve. The vector $\alpha'(t) = (x'(t), y'(t), z'(t)) \in \mathbb{R}^3$ is called the *tangent vector* of α at t. The image set $\alpha(I) \subset \mathbb{R}^3$ is called the *trace* of α.

Let M be a regular surface. A differentiable curve $\alpha\colon I \to M$ is simply a differentiable curve $\alpha\colon I \to \mathbb{R}^3$ such that $\alpha(I) \subset M$, that is, $\alpha(t) \in M$ for every $t \in I$. Fix a point $p \in M$. A vector $v \in \mathbb{R}^3$ is called a *tangent vector* to M at p if there exists a differentiable curve $\alpha\colon (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. The *tangent plane* to M at p is the collection of all tangent vectors to M at p, and it will be denoted by T_pM.

Proposition 1.3.1. For any parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$, one has

$$T_pM = d\varphi(x)(\mathbb{R}^2).$$

Proof. Any vector in the image of $d\varphi(x)$ is of the form $d\varphi(x)\cdot v$, for some $v \in \mathbb{R}^2$ and therefore is the tangent vector at 0 of the differentiable curve $\alpha(t) = \varphi(x+tv)$. Conversely, let $v \in T_pM$, with $v = \alpha'(0)$, where $\alpha\colon (-\epsilon, \epsilon) \to M$ is a differentiable curve, with $\alpha(0) = p$. By virtue of Corollary 1.2.6, the curve $\gamma = \varphi^{-1} \circ \alpha\colon (-\epsilon, \epsilon) \to U$ is differentiable, with $\beta(0) = x$. Since $\alpha = \varphi \circ \beta$, it follows from the chain rule that

$$v = \alpha'(0) = d\varphi(x) \cdot \gamma'(0)$$

lies in the image of $d\varphi(x)$. $\square$

It follows directly from Proposition 1.3.1 that the tangent plane T_pM is a two-dimensional vector subspace of $\mathbb{R}^3$, and it does not depend on the parametrization φ. Moreover, the choice of a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(q)$, determines a basis $\{\varphi_u(p), \varphi_v(p)\}$ of T_pM, called the *basis associated* to φ. Here the notations φ_u, φ_v mean

$$\varphi_u(p) = \frac{\partial \varphi}{\partial u}(q) = d\varphi(q) \cdot e_1 \quad \text{and} \quad \varphi_v(p) = \frac{\partial \varphi}{\partial v}(q) = d\varphi(q) \cdot e_2$$

Let us see how to determine the coordinates of a vector $v \in T_pM$ in the basis $\{\varphi_u(p), \varphi_v(p)\}$ associated to a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(q)$.

Example 1.3.2. Let M be a regular surface given as the inverse image under a differentiable function $f\colon \mathbb{R}^3 \to \mathbb{R}$ of a regular value, namely $M = f^{-1}(c)$. We claim that

$$T_pM = \ker \mathrm{d}f(p)$$

for any $p \in M$. In fact, let $w \in T_pM$, with $w = \alpha'(0)$, where $\alpha\colon (-\epsilon, \epsilon) \to M$ is a diferentiable curve, with $\alpha(0) = p$. Then, $\beta(t) = (f \circ \alpha)(t)$ is a constant curve along $(-\epsilon, \epsilon)$. By the chain rule we obtain

$$0 = \beta'(0) = \mathrm{d}f(p) \cdot w,$$

and this proves the inclusion $T_pM \subset \ker \mathrm{d}f(p)$ and hence the equality by dimensional reasons.

Given a differentiable map $f\colon M \to N$ and a point $p \in M$, we want to define the differential of f at p, denoted by $\mathrm{d}f(p)$, and being a linear map from T_pM to $T_{f(p)}N$. More precisely, for each vector $w \in T_pM$, define

$$\mathrm{d}f(p) \cdot w = (f \circ \alpha)'(0), \tag{1.3}$$

where $\alpha\colon (-\epsilon, \epsilon) \to M$ is a differentiable curve with $\alpha(0) = p$ and $\alpha'(0) = w$.

Proposition 1.3.3. The map $df(p)\colon T_pM \to T_{f(p)}N$ given in (1.3) is well-defined and is linear, and it will be called the *differential* of f at $p \in M$.

Proof. Firstly, we have to check that $df(p) \cdot w$ does not depend on the choice of curve α. Let $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ parametrizations of M and N, respectivaly, with $p = \varphi(q)$ and $f(\varphi(U)) \subset \psi(V)$. Writing

$$\varphi = \varphi(u, v) \quad \text{and} \quad \psi = \psi(z, w),$$

suppose that f is expressed in these coordinates by

$$f(u, v) = (f_1(u, v), f_2(u, v))$$

and that α is expressed by

$$\alpha(t) = (u(t), v(t)).$$

Thus, the curve $\beta = f \circ \alpha$ can be write as

$$\beta(t) = \big(f_1(u(t), v(t)), f_2(u(t), v(t))\big),$$

and the expression of $\beta'(0)$ in the basis $\{\psi_z, \psi_w\}$ is

$$\beta'(0) = \left(\frac{\partial f_1}{\partial u}u'(0) + \frac{\partial f_1}{\partial v}v'(0), \frac{\partial f_2}{\partial u}u'(0) + \frac{\partial f_2}{\partial v}v'(0)\right). \tag{1.4}$$

The relation (1.4) shows that $\beta'(0)$ depends only on the map f and the coordinates $(u'(0), v'(0))$ of w in the basis $\{\varphi_u, \varphi_v\}$. Therefore, $\beta'(0)$ is independent of α. Moreover, it also follows from (1.4) that

$$\beta'(0) = \mathrm{d}f(p) \cdot w = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}. \tag{1.5}$$

This shows that $\mathrm{d}f(p)$ is a linear map from T_pM to $T_{f(p)}N$, whose matrix in the basis $\{\varphi_u, \varphi_v\}$ of T_pM and $\{\psi_z, \psi_w\}$ of $T_{f(p)}N$ is just the matrix given in (1.5). □

Analogously, given a differentiable function $f\colon M \to \mathbb{R}$, we can define the differential of f at $p \in M$ as a linear map $df(p)\colon T_pM \to \mathbb{R}$ by

$$\mathrm{d}f(p) \cdot v = (f \circ \alpha)'(0),$$

where $\alpha\colon (-\epsilon, \epsilon) \to M$ is a differentiable curve with $\alpha(0) = p$ and $\alpha'(0) = v$.

Example 1.3.4. Given a unit vector $v \in \mathbb{R}^3$, consider the height function $f\colon M \to \mathbb{R}$ given by $f(p) = \langle v, p\rangle$, for every $p \in M$. Fix a point $p \in M$ and let $w \in T_pM$. To compute the differential $df(p) \cdot w$, choose a differentiable curve $\alpha\colon (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = w$. Then

$$\begin{aligned} df(p) \cdot w &= (f \circ \alpha)'(0) = \frac{d}{dt} f(\alpha(t))(0) = \frac{d}{dt}\langle v, \alpha(t)\rangle|_{t=0} \\ &= \langle v, \alpha'(0)\rangle = \langle v, w\rangle. \end{aligned}$$

Example 1.3.5. Let $R_\theta\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of angle θ about the axis-z. The rotation $\mathbb{R}_\theta$ is linear, therefore it is differentiable. Moreover, R_θ restricted to the unit sphere $\mathbb{S}^2$ is a differentiable map of $\mathbb{S}^2$. Thus, fixed a point $p \in \mathbb{S}^2$ and given $v \in T_p\mathbb{S}^2$, we obtain

$$dR_\theta(p) \cdot v = R_\theta(v),$$

because R_θ is linear. Note that R_θ leaves the north pole N fixed, and $dR_\theta(N)$ is just a rotation of angle θ in the plane T_NM.

1.4 Orientable surfaces

Intuitively, orientable surfaces are those for which ir is possible to define a clockwise consistently. To ilustrate the underlying idea, we consider two familiar surfaces: a cylinder and a Mobius band. We can distinguish between a cylinder and a Mobius band by noticing that every cylinder has an inside and an outside, and we can paint one blue and other yellow, for example. But if we tray to paint a Mobius band in two colors, we fail because it has just one side.

Let E be a finite-dimensional real vector space. We say that two bases $\mathcal{E}$ and $\mathcal{F}$ *define the same orientation* in E if the transition matrix from $\mathcal{E}$ to $\mathcal{F}$ has positive determinant. In this case, we write $\mathcal{E} \equiv \mathcal{F}$. This property defines an equivalence relation on the set of all bases of E, and each equivalence class according to this relation is called an *orientation* for E. Moreover, the relation $\equiv$ has exactly two equivalence class, that is, the vector space E admits two orientations.

A vector space E together with a choice of orientation $\mathcal{O}$ is called an *oriented vector space*. Once an orientation $\mathcal{O}$ for E is fixed, the other one is called the *opposite orientation*, and it will be denoted by $-\mathcal{O}$. The bases in the orientation $\mathcal{O}$ will be called *positive*, while the others will be called *negative*. A linear isomorphism $T\colon E \to F$ between two oriented vector spaces is called *positive* if maps positive bases of E into positive bases of F.

Example 1.4.1. The Euclidean space $\mathbb{R}^n$ will be considered oriented requiring that the canonical basis of $\mathbb{R}^n$ be positive. Therefore, a linear isomorphism $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is positive if and only if $\det(T) > 0$.

Now we will extend the notion of orientability to each tangent space of a regular surface M. We say that two parametrizations $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ of M are *compatible* if either $\varphi(U) \cap \psi(V) = \emptyset$ or the change of coordinates

$$\psi^{-1} \circ \varphi : \varphi^{-1}(W) \to \psi^{-1}(W),$$

has positive jacobian determinant everywhere on $\varphi^{-1}(W)$.

Remark 1.4.2. If $\varphi(U) \cap \psi(V) \neq \emptyset$, the change of coordinates $\psi^{-1} \circ \varphi$ has jacobian determinant different from zero on $\varphi^{-1}(\varphi(U) \cap \psi(V))$. Since determinant is a continuous function, its sign is constant in each connected component of the open set $\varphi^{-1}(\varphi(U) \cap \psi(V) \subset \mathbb{R}^2$.

Definition 1.4.3. A regular surface M is called *orientable* if there exists a cover $\mathcal{A}$ of M consisting of coordinate neighborhoods such that any two parameterizations of $\mathcal{A}$ are compatible.

The choice of such a cover is called an *orientation* of M, and in this case we say that M is *oriented*. If it is not possible to make such a choice, the surface M is called *nonorientable*.

Example 1.4.4. The plane $\mathbb{R}^2$ is an orientable surface, because the identity map is a global compatible parametrization of $\mathbb{R}^2$. The orientation given by such parametrization is called the *canonical orientation* of $\mathbb{R}^2$.

Example 1.4.5. A regular surface which is the graph of a differentiable function is an orientable surface. More generally, all surfaces which can be covered by one coordinate neighborhood are trivially orientable.

Proposition 1.4.6. *An orientation on a regular surface M determines an orientation on each tangent plane of M.*

Proof. Let $\mathcal{A}$ be an orientation on M. Given a point $p \in M$, consider a parametrization $\varphi \in \mathcal{A}$, with $p = \varphi(x)$, and define an orientation $\mathcal{O}_p$ on T_pM requiring the basis $\{d\varphi(x) \cdot e_1, d\varphi(x) \cdot e_2\}$ be positive. If ψ is another parametrization in the cover $\mathcal{A}$, with $p = \psi(y)$, we obtain:

$$\mathrm{d}\psi(y) = \mathrm{d}(\varphi \circ \varphi^{-1} \circ \psi)(y) = \mathrm{d}\varphi(x) \circ \mathrm{d}(\varphi^{-1} \circ \psi)(y).$$

The isomorphism $d(\varphi^{-1} \circ \psi)(y)$ preserve orientation, because φ and ψ are compatible, e $d\varphi(x)$ preserves orientation by hypothesis. Therefore, the set $\{d\psi(y) \cdot e_1, d\psi(y) \cdot e_2\}$ is also a positive basis of T_pM. $\square$

Before giving a geometric interpretation of the idea of orientability of a regular surface in $\mathbb{R}^3$, we need a few definitions.

Given a regular surface M, the inner product of $\mathbb{R}^3$ naturally induces an inner product in each tangent plane T_pM of M. More precisely, given $p \in M$ and $v, w \in T_pM$, we define $\langle v, w \rangle$ to be the inner product of v and w as vectors of $\mathbb{R}^3$.

The inner product defined above allows us to consider the notion of orthogonality. More precisely, we say that a vector $\eta \in \mathbb{R}^3$ is *orthogonal* to a regular surface M at a point $p \in M$ if $\langle \eta, v \rangle = 0$, for every $v \in T_pM$. Globally, a *normal vector field* to a regular surface M is a map $\eta\colon M \to \mathbb{R}^3$ such that $\eta(p)$ is orthogonal to T_pM, for every $p \in M$.

Theorem 1.4.7. *A regular surface $M \subset \mathbb{R}^3$ is orientable if and only if there exists a differentiable unit normal vector field $N\colon M \to \mathbb{R}^3$ on M.*

Proof. Let $\mathcal{A}$ be an orientation of M, that is, a cover of M by compatible coordinate neighborhoods. Fix a point $p \in M$ with $p = \varphi(u, v)$, where $\varphi\colon U \to \varphi(U)$ is a positive parametrization of M. Now, define a map $N\colon \varphi(U) \to \mathbb{R}^3$ by

$$N(q) = \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|}(q), \tag{1.6}$$

for every $q \in \varphi(U)$, where $\times$ denotes the vector product in $\mathbb{R}^3$. Thus we obtain a differentiable map, orthogonal to M in every point of $\varphi(U)$. If $\psi\colon V \to \psi(V)$ is another positive parametrization of M with $p = \psi(z, w)$, let us denote by $h = \varphi^{-1} \circ \psi$ the change of coordinates, with $(u, v) = h(z, w)$. Thus $\psi = \varphi \circ h$ and by setting $q = (z, w)$, we obtain

$$\begin{aligned} \psi_z(q) &= \varphi_u(h(q))\frac{\partial u}{\partial z}(q) + \varphi_v(h(q))\frac{\partial v}{\partial z}(q) \\ \psi_w(q) &= \varphi_u(h(q))\frac{\partial u}{\partial w}(q) + \varphi_v(h(q))\frac{\partial v}{\partial w}(q) \end{aligned}.$$

It follows that

$$\psi_z \times \psi_w = \det(dh(q)) \cdot \varphi_u \times \varphi_v,$$

and from which we conclude that the normals associated to φ and ψ coincide, since $\det(dh(q)) > 0$. Therefore, if p belongs to the intersection of two coordinate neighborhoods, the corresponding normals coincide, and thus we obtain an unit normal differentiable vector field $N\colon M \to \mathbb{R}^3$. Conversely, given a point $p \in M$, define an orientation in T_pM as follows: a basis $\{v_1, v_2\}$ of T_pM is positive if and only if $\{v_1, v_2, N(p)\}$ is a positive basis of $\mathbb{R}^3$. Given a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p = \varphi(x)$ and U connected, and changing the sign of φ if necessary, we can assume that the set

$$\{d\varphi(x) \cdot e_1, d\varphi(x) \cdot e_2, N(\varphi(x))\}$$

is a positive basis of $\mathbb{R}^3$, for every $x \in U$. Thus, for each point $p \in M$, we can choose a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p \in \varphi(U)$, such that $d\varphi(x)\colon \mathbb{R}^2 \to T_{\varphi(x)}M$ is a positive linear isomorphism, for every $x \in U$. Denote by $\mathcal{A}$ the cover of M by such parametrizations. If $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ are two parametrizations in $\mathcal{A}$, with $\varphi(U) \cap \psi(V) \neq \emptyset$, then $\psi^{-1} \circ \varphi$ has positive jacobian determinant everywhere, since $d(\psi^{-1} \circ \varphi)(x)$ is the composite of two positive linear isomorphisms. $\square$

It follows from the proof of Theorem 1.4.7 that, given a point $p \in M$ and a parametrization $\varphi\colon U \to \varphi(U)$ of M, with $p \in \varphi(U)$, we can always consider an unit normal differentiable vector field N in a neighborhood of p and given by (1.6). Thus, any regular surface M is always locally orientable.

Let us now look at some examples of orientable surfaces.

Example 1.4.8. Let $\mathcal{P}$ be the plane plane through the point $p \in \mathbb{R}^3$ which contains the orthonormal vectors $w_1, w_2 \in \mathbb{R}^3$. Thus, a parametrization of $\mathcal{P}$ is given by

$$\varphi(u, v) = p + uw_1 + vw_2,$$

with $(u, v) \in \mathbb{R}^2$. In this case one has $\varphi_u = w_1$ and $\varphi_v = w_2$, therefore $N = w_1 \times w_2$ is an unit normal vector field along $\mathcal{P}$.

Example 1.4.9. Consider the right cylinder $\mathcal{C}$ over the circle $x^2 + y^2 = 1$. Then $\mathcal{C}$ admits a parametrization $\varphi\colon U \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (\cos u \sin u, v),$$

where

$$U = \{(u, v) \in \mathbb{R}^2 : 0 < u < 2\pi \text{ and } v \in \mathbb{R}\}.$$

In this case, we obtain

$$\varphi_u = (-\sin u, \cos u, 0) \quad \text{and} \quad \varphi_v = (0,0,1),$$

which implies that

$$N = \varphi_u \times \varphi_v = (\cos u, \sin u, 0)$$

is an unit normal vector field to the cylinder $\mathcal{C}$.

Example 1.4.10. The tangent plane to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ at a point $p \in \mathbb{S}^2$ is given by

$$T_p\mathbb{S}^2 = p^\perp,$$

where $p^\perp = \{v \in \mathbb{R}^3 : \langle v, p\rangle = 0\}$. In fact, fix a vector $v \in T_p\mathbb{S}^2$ with $v = \alpha'(0)$, where $\alpha\colon (-\epsilon, \epsilon) \to \mathbb{S}^2$ is a differentiable curve, with $\alpha(0) = p$. Since $\alpha(t) \in \mathbb{S}^2$ for every $t \in (-\epsilon, \epsilon)$, one has $\|\alpha(t)\| = 1$, for every $t \in (-\epsilon, \epsilon)$. This implies that

$$2\langle \alpha'(t), \alpha(t)\rangle = 0,$$

for every $t \in (-\epsilon, \epsilon)$. For $t = 0$, we obtain $\langle v, p\rangle = 0$. This shows that $T_p\mathbb{S}^2 \subset p^\perp$ and hence the equality by dimensional reasons. Therefore, the position vector field $N\colon \mathbb{S}^2 \to \mathbb{R}^3$, $N(p) = p$, is an unit normal vector field to the sphere, and this shows that $\mathbb{S}^2$ is orientable.

Example 1.4.11. Let M be a regular surface given as the inverse image under a differentiable function $f\colon \mathbb{R}^3 \to \mathbb{R}$ of a regular value $c \in \mathbb{R}$, that is, $M = f^{-1}(c)$. Fix a point $p \in M$, with $p = (x_0, y_0, z_0)$, and consider a differentiable parametrized curve $\alpha\colon (-\epsilon, \epsilon) \to M$, with $\alpha(0) = p$ and

$$\alpha(t) = (x(t), y(t), z(t)).$$

Since $\alpha(t) \in M$ for every $t \in (-\epsilon, \epsilon)$, we obtain

$$f(\alpha(t)) = c, \tag{1.7}$$

for every $t \in (-\epsilon, \epsilon)$. By differentiating both sides of (1.7) with respect to t, we see that at $t = 0$

$$\begin{aligned} 0 &= \frac{d}{dt} f(\alpha(t))(0) = \frac{\partial f}{\partial x}(p) \cdot \frac{dx}{dt}(0) + \frac{\partial f}{\partial y}(p) \cdot \frac{dy}{dt}(0) + \frac{\partial f}{\partial z}(p) \cdot \frac{dz}{dt}(0) \\ &= \left\langle \left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p) \right), \left(\frac{dx}{dt}(0), \frac{dy}{dt}(0), \frac{dz}{dt}(0) \right) \right\rangle \\ &= \langle \operatorname{grad} f(p), \alpha'(0)\rangle = \langle \operatorname{grad} f(p), v\rangle. \end{aligned}$$

This shows that the gradient vector of f at $p \in M$ is orthogonal to T_pM. Therefore, the map

$$N(p) = \frac{\text{grad} f}{\|\text{grad} f\|}(p)$$

is an unit normal vector field to M, and thus M is orientable.

Chapter 2

The local theory of surfaces

2.1 Derivatives of vector fields in $\mathbb{R}^3$

A *vector field* defined in an open set $U \subset \mathbb{R}^3$ is simply a differentiable map $X: U \to \mathbb{R}^3$ that maps each point $p \in U$ in a vector $X(p) \in \mathbb{R}^3$. The set of all vector fields defined in the open $U \subset \mathbb{R}^3$ will be denoted by $\mathfrak{X}(U)$. Under the natural operations

$$\begin{aligned} (X+Y)(p) &= X(p) + Y(p) \\ (c \cdot X)(p) &= c \cdot X(p) \end{aligned},$$

this set $\mathfrak{X}(U)$ is a real vector space. The next question we want to address is to define a notion of derivative of vector fields along a vector, and more generally, along another vector field.

Let $Y \in \mathfrak{X}(U)$ be a vector field and fix a point $p \in U$. The *directional derivative* of Y at p, along a vector $v \in \mathbb{R}^3$, is defined by

$$\widetilde{\nabla}_v Y = dY(p) \cdot v. \tag{2.1}$$

Note that if $\alpha: (-\epsilon, \epsilon) \to U$ is a differentiable curve with $\alpha(0) = p$ and $\alpha'(0) = v$, then

$$\begin{aligned} \lim_{t \to 0} \frac{Y(\alpha(t)) - Y(\alpha(0))}{t} &= \frac{d}{dt} Y(\alpha(t))|_{t=0} \\ &= dY(p) \cdot v \\ &= \widetilde{\nabla}_v Y. \end{aligned}$$

This implies that the vector $\widetilde{\nabla}_v Y$ depends only on the values of Y along a differentiable curve through p with velocity v. We can extend the above definition. More precisely, given two vector fields $X, Y \in \mathfrak{X}(U)$, we will define another vector field $\widetilde{\nabla}_X Y$ by

$$\left(\widetilde{\nabla}_X Y\right)(p) = \widetilde{\nabla}_{X(p)} Y. \tag{2.2}$$

In the same way as in (2.1), the left-hand side of (2.2) depends only on the values of Y along a differentiable curve through p with velocity $X(p)$.

To understand more fully what the derivative in (2.1) means, it is useful to bear in mind some terminology. Fix a point $p \in U$. Given a vector $v \in \mathbb{R}^3$, we can define a natural function $v\colon \mathrm{C}^\infty(U) \to \mathbb{R}$ by

$$v(f) = \frac{d}{dt} f(\alpha(t))|_{t=0} = \frac{\partial f}{\partial v}(p) = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(p) \cdot v_i, \tag{2.3}$$

for every $f \in \mathrm{C}^\infty(U)$, where $\alpha\colon (-\epsilon, \epsilon) \to U$ is a differentiable curve with $\alpha(0) = p$ and $\alpha'(0) = v$, and $v = (v_1, v_2, v_3)$. The function given in (2.3) is called a *derivation* at p along the vector v. On the other hand, given a vector field $X \in \mathfrak{X}(U)$ and a function $f \in \mathrm{C}^\infty(U)$, we can define another function $X(f)\colon U \to \mathbb{R}$ by

$$X(f)(p) = X(p)(f) = df(p) \cdot X(p),$$

where $X(p)(f)$ is the derivation at p along the vetor $X(p)$ given in (2.3).

Therefore, motivated by these considerations, fixed a point $p \in U$ and given a vector field $Y \in \mathfrak{X}(U)$, with $Y = (Y_1, Y_2, Y_3)$, we can write

$$\begin{aligned} \widetilde{\nabla}_v Y &= \frac{d}{dt} Y(\alpha(t))|_{t=0} = \frac{d}{dt}\big(Y_1(\alpha(t)), Y_2(\alpha(t)), Y_2(\alpha(t))\big)|_{t=0} \\ &= \big(v(Y_1), v(Y_2), v(Y_3)\big), \end{aligned} \tag{2.4}$$

where $\alpha\colon (-\epsilon, \epsilon) \to U$ is a differentiable curve with $\alpha(0) = p$ and $\alpha'(0) = v$. In other words, the directional derivative of a vector field along a vector is just a derivation of its coordinates functions related to that vector.

In the sequel, we will define a map

$$\widetilde{\nabla}\colon \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U) \tag{2.5}$$

that assigns to each pair $X, Y \in \mathfrak{X}(U)$ the vector field $\widetilde{\nabla}_X Y \in \mathfrak{X}(U)$ given in (2.2). The map $\widetilde{\nabla}$ defined in (2.5) is called the *derivative* of the vector fields in the open set $U \subset \mathbb{R}^3$.

Proposition 2.1.1. The map $\widetilde{\nabla}$ satisfies the following basic properties:

(a) $\widetilde{\nabla}_{fX}Y = f\widetilde{\nabla}_X Y$,

(b) $\widetilde{\nabla}_X fY = X(f)Y + f\widetilde{\nabla}_X Y$,

for every $X, Y \in \mathfrak{X}(U)$ and $f \in \mathrm{C}^\infty(U)$.

Proof. Fix a point $p \in U$. By writing $Y = (Y_1, Y_2, Y_3)$, we obtain

$$\begin{aligned}
\left(\widetilde{\nabla}_{fX}Y\right)(p) &= (f(p)X_pY_1, f(p)X_pY_2, f(p)X_pY_3) \\
&= f(p)(X_pY_1, X_pY_2, X_pY_3) \\
&= f(p)\left(\widetilde{\nabla}_X Y\right)(p)
\end{aligned}$$

and

$$\begin{aligned}
\left(\widetilde{\nabla}_X fY\right)(p) &= (X_p(fY_1), X_p(fY_2), X_p(fY_3)) \\
&= \big(X(f)Y_1 + fX(Y_1), \ldots, X(f)Y_3 + fX(Y_3)\big)(p) \\
&= X_p(f)(Y_1, Y_2, Y_3) + f(p)\widetilde{\nabla}_{X_p} Y \\
&= \left(X(f)Y + f\widetilde{\nabla}_X Y\right)(p),
\end{aligned}$$

and this concludes the proof. $\square$

Proposition 2.1.2. Given $X, Y \in \mathfrak{X}(U)$, there exists a unique vector field $Z \in \mathfrak{X}(U)$ such that

$$Z(f) = (XY - YX)(f), \tag{2.6}$$

for every $f \in \mathrm{C}^\infty(U)$.

Proof. Writing $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$, we obtain

$$XY(f) = X\left(\sum_{j=1}^{3} Y_j \frac{\partial f}{\partial x_j}\right) = \sum_{i,j=1}^{3}\left(X_i \frac{\partial Y_j}{\partial x_i}\frac{\partial f}{\partial x_j} + X_iY_j \frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

and

$$XY(f) = Y\left(\sum_{i=1}^{3} X_i \frac{\partial f}{\partial x_i}\right) = \sum_{i,j=1}^{3}\left(Y_j \frac{\partial X_i}{\partial x_j}\frac{\partial f}{\partial x_i} + X_iY_j \frac{\partial^2 f}{\partial x_j \partial x_i}\right).$$

Thus,

$$(XY - YX)(f) = \sum_{i,j=1}^{3} \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial f}{\partial x_j}. \tag{2.7}$$

Therefore, by defining the vector field $Z = (Z_1, Z_2, Z_3)$ by

$$Z_j = \sum_{i=1}^{3} \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right),$$

it follows that Z is the unique vector field in U satisfying (2.6). □

The vector field Z, given by Proposition 2.1.2, is called the *Lie bracket* of X and Y, and it will be denoted by $[X, Y]$.

Proposition 2.1.3. The map $\widetilde{\nabla}$ satisfies the following properties:

(a) $X\langle Y, Z\rangle = \langle \widetilde{\nabla}_X Y, Z\rangle + \langle Y, \widetilde{\nabla}_X Z\rangle$,

(b) $\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X = [X, Y]$,

for every $X, Y, Z \in \mathfrak{X}(U)$. Conversely, if

$$\overline{\nabla} : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$$

is a map that satisfies the properties (a) and (b) above, then $\overline{\nabla} = \widetilde{\nabla}$.

Proof. Part (a) follows from

$$\begin{aligned} X\langle Y, Z\rangle(p) &= X(p) \left(\sum_{i=1}^{3} Y_i Z_i \right) \\ &= \sum_{i=1}^{3} \big(X(p)(Y_i) Z_i + Y_i X(p)(Z_i) \big) \\ &= \langle \widetilde{\nabla}_{X(p)} Y, Z(p)\rangle + \langle Y(p), \widetilde{\nabla}_{X(p)} Z\rangle, \end{aligned}$$

for every $p \in U$. For part (b), writing $\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X = (W_1, W_2, W_3)$ and using (2.4), we obtain

$$W_i = X(p)(Y_i) - Y(p)(X_i) = \sum_{j=1}^{3} \left(X_j \frac{\partial Y_i}{\partial x_j}(p) - Y_j \frac{\partial X_i}{\partial x_j}(p) \right),$$

and the formula (b) follows from (2.7). Conversely, suppose that a map $\overline{\nabla} : \mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ satisfies the properties (a) and (b). By part (a), one has

$$\begin{aligned} X\langle Y, Z\rangle &= \left\langle \overline{\nabla}_X Y, Z\right\rangle + \left\langle Y, \overline{\nabla}_X Z\right\rangle && (2.8)\\ Y\langle X, Z\rangle &= \left\langle \overline{\nabla}_Y X, Z\right\rangle + \left\langle X, \overline{\nabla}_Y Z\right\rangle && (2.9)\\ Z\langle X, Y\rangle &= \left\langle \overline{\nabla}_Z X, Y\right\rangle + \left\langle X, \overline{\nabla}_Z Y\right\rangle, && (2.10) \end{aligned}$$

for every $X, Y, Z \in \mathfrak{X}(U)$. Adding (2.8) and (2.9), subtracting (2.10) and using part (b), we obtain

$$\begin{aligned} X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle =& \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle \\ &+ \langle [X, Y], Z\rangle + 2\left\langle \overline{\nabla}_Y X, Z\right\rangle \end{aligned}$$

that is,

$$\begin{aligned} 2\left\langle \overline{\nabla}_Y X, Z\right\rangle =& X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle [X, Z], Y\rangle \\ &- \langle [Y, Z], X\rangle - \langle [X, Y], Z\rangle \end{aligned} \tag{2.11}$$

The equation (2.11) shows that $\overline{\nabla}$ is uniquely determined by the inner product $\langle , \rangle$, and therefore

$$\left\langle \overline{\nabla}_Y X, Z\right\rangle = \left\langle \widetilde{\nabla}_Y X, Z\right\rangle,$$

for every $X, Y, Z \in \mathfrak{X}(U)$, and this shows that $\overline{\nabla} = \widetilde{\nabla}$. $\square$

Proposition 2.1.4. For any vector fields $X, Y, Z \in \mathfrak{X}(U)$, the following holds:

$$\widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z = 0. \tag{2.12}$$

Proof. By definition, we have

$$\widetilde{\nabla}_X \widetilde{\nabla}_Y Z = \widetilde{\nabla}_X (YZ_1, YZ_2, YZ_3) = (XYZ_1, XYZ_2, XYZ_3)$$

and

$$\widetilde{\nabla}_Y \widetilde{\nabla}_X Z = \widetilde{\nabla}_Y (XZ_1, XZ_2, XZ_3) = (YXZ_1, YXZ_2, YXZ_3).$$

This implies that

$$\begin{aligned} \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z &= (XYZ_1 - YXZ_1, XYZ_2 - YXZ_2, XYZ_3 - YXZ_3) \\ &= \big([X, Y]Z_1, [X, Y]Z_2, [X, Y]Z_3\big) \\ &= \widetilde{\nabla}_{[X,Y]} Z, \end{aligned}$$

and this proves equation (2.12). $\square$

2.2 Derivatives of vector fields on surfaces

A *vector field along* a regular surface M in $\mathbb{R}^3$ is just a differentiable map $X: M \to \mathbb{R}^3$ that maps each point $p \in M$ in a vector $X(p) \in \mathbb{R}^3$. We say that X is *tangent* to M if $X(p) \in T_pM$ for every $p \in M$. We also say that X is *normal* to M if $X(p) \perp T_pM$ for every $p \in M$. We denote by $\mathfrak{X}(M)$ the set of all tangent vector fields to M. In a similar way to the case of vector fields in $\mathbb{R}^3$, the set $\mathfrak{X}(M)$ becomes a real vector space endowed with the natural operations.

Given two vector fields $X, Y \in \mathfrak{X}(M)$, we define a map $[X, Y]: M \to \mathbb{R}^3$ by

$$[X, Y](p) = [\widetilde{X}, \widetilde{Y}](p) \tag{2.13}$$

for every $p \in M$, where $\widetilde{X}$, $\widetilde{Y}$ are differentiable extensions of X and Y, respectively, to an open set $U \subset \mathbb{R}^3$, with $p \in U$. We will see that (2.13) is well defined and $[X, Y] \in \mathfrak{X}(M)$ for any choice of vector fields $X, Y \in \mathfrak{X}(M)$.

Given a parametrization $\varphi: U \to \varphi(U)$ of M, we can write on the coordinate neighborhood $\varphi(U)$

$$X = \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_{j=1}^{2} b_j \frac{\partial}{\partial x_j},$$

where $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ are the coordinate vector fields associated to φ.

Proposition 2.2.1. On the coordinate neighborhood $\varphi(U)$, we have

$$[X, Y] = \sum_{i,j=1}^{2} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}. \tag{2.14}$$

Proof. Consider the map $\phi: U \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ given by

$$\phi(x_1, x_2, x_3) = \varphi(x_1, x_2) + x_3 N(\varphi(x_1, x_2)),$$

where N is an unit normal vector field to M along $\varphi(U)$. Note that φ is differentiable and

$$\begin{aligned}
\frac{\partial \phi}{\partial x_1} &= \frac{\partial \varphi}{\partial x_1} + x_3 \frac{\partial N}{\partial x_1}, \\
\frac{\partial \phi}{\partial x_2} &= \frac{\partial \varphi}{\partial x_2} + x_3 \frac{\partial N}{\partial x_2}, \\
\frac{\partial \phi}{\partial x_3} &= N.
\end{aligned}$$

Since $\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, N$ are linearly independent, the same is true for $\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}$, for sufficiently small $\epsilon > 0$. Restricting the range $U \times (-\epsilon, \epsilon)$ if necessary, it follows from the inverse function theorem that ϕ is a diffeomorphism between $U \times (-\epsilon, \epsilon)$ and $\phi(U \times (-\epsilon, \epsilon))$. Thus, if

$$\widetilde{X} = \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \widetilde{Y} = \sum_{j=1}^{3} b_j \frac{\partial}{\partial x_j}$$

are differentiable extensions of $X|_{\varphi(U)}$ and $Y|_{\varphi(U)}$, respectivaly, it follows from (2.7) that

$$[\widetilde{X}, \widetilde{Y}] = \sum_{i,j=1}^{3} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

Therefore,

$$[\widetilde{X}, \widetilde{Y}]|_{\varphi(U)} = \sum_{i,j=1}^{2} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j},$$

and this concludes the proof. □

Similarly as vector fields in $\mathbb{R}^3$, the vector field $[X, Y] \in \mathfrak{X}(M)$ will be called the *Lie bracket* of X and Y. It follows directly from (2.14) the following

Corollary 2.2.2. If $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ are the coordinate vector fields associated to a parametrization φ of M, then

$$\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right] = 0.$$

Example 2.2.3. In $\mathbb{R}^2$, with canonical coordinates (x, y), consider the vector fields $X = y\frac{\partial}{\partial y}$ and $Y = x\frac{\partial}{\partial y}$. Given a function $f \in C^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned} [X, Y](f) &= \left[y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y} \right](f) \\ &= yx\frac{\partial^2 f}{\partial y^2} - x\frac{\partial f}{\partial y} - xy\frac{\partial^2 f}{\partial y^2} \\ &= -x\frac{\partial}{\partial y}(f) = -Y(f). \end{aligned}$$

Therefore, in this case one has $[X, Y] = -Y$.

Let $f\colon M \to N$ be a diffeomorphism between two regular surfaces M and N. Given a vector field $X \in \mathfrak{X}(M)$, denote by f_*X the vector field in N given by

$$(f_*X)(f(p)) = \mathrm{d}f(p) \cdot X(p),$$

for every $p \in M$.

Corollary 2.2.4. If $f\colon M \to N$ is a diffeomorphism, then

$$[f_*X, f_*Y] = f_*[X, Y], \tag{2.15}$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. Given a parametrization $\varphi\colon U \to \varphi(U)$ of M, we can write X and Y along $\varphi(U)$ as

$$X = \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_{j=1}^{2} b_j \frac{\partial}{\partial x_j}.$$

Let $\psi\colon U \to \psi(V)$ be the map given $\psi = f \circ \varphi$, where $V = f(\varphi(U))$. Since φ is a parametrization of M and f is a diffeomorphism, it follows that ψ is a parametrization of N, and the coordinate vector fields $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}$ associated to ψ are given by

$$\frac{\partial}{\partial y_i} = \psi_* \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq 2.$$

Thus, along $\psi(V)$ we have

$$\psi_*X = \sum_{i=1}^{2} \widetilde{a}_i \frac{\partial}{\partial y_i} \quad \text{and} \quad \psi_*Y = \sum_{j=1}^{2} \widetilde{b}_j \frac{\partial}{\partial y_j},$$

where $a_i \circ \psi = \widetilde{a}_i$ and $b_j \circ \psi = \widetilde{b}_j$, and equation (2.15) follows from (2.14). □

Given a tangent vector field $X \in \mathfrak{X}(M)$ and a vector field Y along M, define

$$\left(\widetilde{\nabla}_X Y\right)(p) = \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}\right)(p), \tag{2.16}$$

for every $p \in M$, where $\widetilde{X}, \widetilde{Y}$ are differentiable extensions of X and Y, respectively. Since the right hand side of (2.16) does not depend on the choice of the extensions $\widetilde{X}$ and $\widetilde{Y}$, the vector field $\widetilde{\nabla}_X Y$ is well defined. Given $X, Y \in \mathfrak{X}(M)$, we define a vector field $\nabla_X Y$ by

$$(\nabla_X Y)(p) = \left(\widetilde{\nabla}_X Y\right)^T (p), \tag{2.17}$$

where the right hand side of (2.17) denotes the tangent component of $\widetilde{\nabla}_X Y$ along the surface M.

Proposition 2.2.5. The map $\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given in (2.17) satisfies the following properties:

(a) $\nabla_{fX} Y = f \nabla_X Y$,

(b) $\nabla_X fY = X(f)Y + f\nabla_X Y$,

(c) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$,

(d) $\nabla_X Y - \nabla_Y Z = [X, Y]$,

for all $X, Y, Z \in \mathfrak{X}(M)$ and every $f \in \mathrm{C}^\infty(N)$. Conversely, if a map $\overline{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfies properties (a)–(d) above, then $\overline{\nabla} = \nabla$.

Proof. Parts (a) and (b) follow by taking the tangent component of formulas (a) and (b), respectively, of Proposition 2.1.1. Analogously, part (c) follows by taking the tangent component of formula (c) of Proposition 2.1.3. Part (d) follows directly of formula (d) of Proposition 2.1.3. The converse is entirely analogous to Proposition 2.1.3. □

Corollary 2.2.6. Let $f \colon M \to N$ be a local isometry between two regular surfaces M and N. Then

$$f_* \left(\nabla_X Y\right) = \nabla_{f_* X} f_* Y,$$

for every $X, Y \in \mathfrak{X}(M)$.

Proof. Define a map $\overline{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$\overline{\nabla}_X Y = f_*^{-1} \left(\nabla_{f_* X} f_* Y\right).$$

A straightforward computation shows that $\overline{\nabla}$ satisfies properties (a)–(d) of Proposition 2.2.5. For example, property (d) follows from

$$\begin{aligned} \overline{\nabla}_X Y - \overline{\nabla}_Y X &= f_*^{-1}\big(\nabla_{f_* X} f_* Y - \nabla_{f_* Y} f_* X\big) \\ &= f_*^{-1}\big([f_* X, f_* Y]\big) \\ &= f_*^{-1}\big(f_*[X, Y]\big) \\ &= [X, Y], \end{aligned}$$

using Corollary 2.2.4. Therefore, by the uniqueness of derivative it follows that $\overline{\nabla}$ coincides with the derivative ∇ of M. □

2.3 The Gauss map of a regular surface

Let M be an orientable regular surface. This means, according to Theorem 1.4.7, that there exists a differentiable unit vector field $N\colon M \to \mathbb{R}^3$, which is normal along M and called an *orientation* for M. Fixed an orientation N of M, note that the map $N\colon M \to \mathbb{R}^3$ maps each point $p \in M$ into a vector $N(p)$ in the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. The map

$$N\colon M \to \mathbb{S}^2,$$

thus defined, is called the *Gauss map* of the surface M.

The Gauss map is differentiable and its differential $dN(p)$ at a point $p \in M$ is a linear map from T_pM to $T_{N(p)}\mathbb{S}^2$. Since the planes T_pM and $T_{N(p)}\mathbb{S}^2$ are parallel, because they have the same normal vector $N(p)$, we can identify them and, thus, $dN(p)$ is a linear operator on T_pM.

From a variational point of view, the differential $dN(p)$ measures the rate of change of the normal vector N in relation to the tangent plane T_pM in a neighborhood of p. In fact, given a vector $v \in T_pM$, let $\lambda\colon (-\epsilon, \epsilon) \to M$ be a differentiable curve, with $\lambda(0) = p$ and $\lambda'(0) = v$, and consider the restriction $N(t) = N(\lambda(t))$, which is a differentiable curve in the sphere $\mathbb{S}^2$. Since $\|N\| = 1$, the vector $N'(0) = dN(p) \cdot v$ is tangent to M at p, and it measures the rate of change of N, restricted to the curve λ, at $t = 0$.

We will make some considerations in order to obtain a geometric interpretation for the differential of the Gauss map. Given two vector fields $X, Y \in \mathfrak{X}(M)$, define

$$\alpha(X, Y) = \left(\widetilde{\nabla}_X Y\right)^{\perp}, \tag{2.18}$$

where the right-hand side of (2.18) denotes the normal component of $\widetilde{\nabla}_X Y$ along M. Taking normal components in the equation

$$\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X = [X, Y],$$

and using the fact that $[X, Y] \in \mathfrak{X}(M)$, we obtain

$$\alpha(X, Y) = \alpha(Y, X),$$

for every $X, Y \in \mathfrak{X}(M)$. Since $\left(\widetilde{\nabla}_X Y\right)(p)$ depends only of $X(p)$, it follows from the symmetry of α that $\alpha(X, Y)(p)$ depends only of $X(p)$ e $Y(p)$. Thus, for each point $p \in M$, we obtain a symmetric bilinear map

$$\alpha_p : T_pM \times T_pM \to T_pM^{\perp}$$

given by $\alpha_p(X_p, Y_p) = \alpha(X, Y)(p)$. The bilinear map α_p is usually called the *second fundamental form* of M at p. Such map defines a unique linear operator $A\colon T_pM \to T_pM$ by the relation

$$\langle AX, Y\rangle = \langle \alpha_p(X, Y), N(p)\rangle,$$

for every $X, Y \in T_pM$. The symmetry of α implies that A is a self-adjoint linear operator, and it will be called the *shape operator* of M at p.

Proposition 2.3.1. Fixed a point $p \in M$, we have

$$AX = -\widetilde{\nabla}_X N,$$

for every $X \in T_pM$.

Proof. Given $X, Y \in T_pM$, one has

$$\begin{aligned} \langle AX, Y\rangle &= \langle \alpha_p(X, Y), N(p)\rangle \\ &= \left\langle \widetilde{\nabla}_X Y, N(p)\right\rangle \\ &= -\left\langle \widetilde{\nabla}_X N, Y\right\rangle. \end{aligned}$$

Since $Y \in T_pM$ is arbitrary, this completes the proof. $\square$

From Proposition 2.3.1, we identify the shape operator $A\colon T_pM \to T_pM$ with the differential $dN(p)$ of the Gauss map at the point $p \in M$. Since A is self-adjoint, it follows from the spectral theorem that there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM such that

$$Ae_i = k_i e_i, \quad 1 \le i \le 2. \tag{2.19}$$

Moreover, the real numbers k_1 and k_2 ($k_1 \ge k_2$) are the maximum and minimum, respectively, of the quadratic form II_p defined in T_pM by

$$\mathrm{II}_p(v) = -\langle Av, v\rangle, \tag{2.20}$$

restricted to the unit circle of T_pM.

Example 2.3.2. Consider a plane $\mathcal{P}$ in $\mathbb{R}^3$ given by $ax + by + cz + d = 0$. An unit vector field N normal to $\mathcal{P}$ is given by

$$N = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}},$$

which is constant. This implies that the shape operator $A = -dN(p)$ vanishes identically at any point p.

Example 2.3.3. In the unit sphere $\mathbb{S}^2$, we fix an orientation by choosing $N(p) = -p$ as a normal vector field. Thus, the shape operator is

$$Av = -dN(p) \cdot v = v,$$

for every $v \in T_p\mathbb{S}^2$.

Example 2.3.4. In the cylinder

$$\mathcal{C} = \{(x, y, z \in \mathbb{R}^3 : x^2 + y^2 = 1)\},$$

we fix an orientation by choosing $N = (-x, -y, 0)$ as a normal vector field. Thus, given $p \in \mathcal{C}$ and $v \in T_p\mathcal{C}$, with $v = (v_1, v_2, v_3)$, one has

$$Av = -dN(p) \cdot v = (v_1, v_2, 0).$$

In particular, if $v \in T_p\mathcal{C}$ is parallel to the axis-z, then $Av = 0 \cdot v$; if $w \in T_p\mathcal{C}$ is parallel to the plane-xy, then $Aw = 1 \cdot w$. It follows that the vectors v and w are eigenvectors of A with eigenvalues 0 and 1, respectively.

Example 2.3.5. A parametrization for the hyperbolic paraboloid

$$\mathcal{H} = \{(x, y, z) \in \mathbb{R}^3 : z = y^2 - x^2\}$$

is given by

$$\varphi(x, y) = (x, y, y^2 - x^2).$$

Thus, the coordinate vector fields are

$$\frac{\partial \varphi}{\partial x} = (1, 0, -2x) \quad \text{and} \quad \frac{\partial \varphi}{\partial y} = (0, 1, 2y),$$

and an unit vector field N normal to $\mathcal{H}$ is

$$N(x, y) = \frac{1}{\sqrt{x^2 + y^2 + 1/4}}(x, -y, 1/2). \tag{2.21}$$

At the origem $p = (0, 0, 0)$, the tangent plane $T_p\mathcal{H}$ coincides with the plane-xy, since $\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}$ agree with the unit canonical vectors fields e_1 and e_2, respectively. Given $v \in T_p\mathcal{H}$, let $\lambda\colon (-\epsilon, \epsilon) \to \mathcal{H}$ be a differentiable curve with $\lambda(0) = p$ and $\lambda'(0) = v$. If $\lambda(t) = (x(t), y(t), z(t))$, then

$$\lambda'(0) = (x'(0), y'(0), 0).$$

If $N(t) = N(\lambda(t))$, it follows from (2.21) that

$$N'(0) = 2(x'(0), -y'(0), 0).$$

Thus, at the origem $p = (0, 0, 0)$, the shape operator A is given by

$$Av = 2(-v_1, v_2, 0),$$

where $v = (v_1, v_2, v_3)$. This implies that the canonical vectors $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ are eigenvectors of A, with eigenvalues -2 and 2, respectively.

2.4 Euler's principal curvatures

A parametrized differentiable curve $\lambda\colon I \to \mathbb{R}^3$ is said to be *regular* $\lambda'(t) \neq 0$ for all $t \in I$. Fixed an instant $t_0 \in I$, the *arc length* of λ, from the instant t_0, is the function

$$s(t) = \int_{t_0}^{t} \|\lambda'(t)\| dt. \tag{2.22}$$

We will say that λ is *parameterized by arc length* if for every $t_1, t_2 \in I$, with $t_1 \leq t_2$, the arc length of λ from t_1 to t_2 is equal to $t_2 - t_1$, that is

$$\int_{t_0}^{t} \|\lambda'(t)\| dt = t_2 - t_1. \tag{2.23}$$

Note that, if $\|\lambda'(t)\| = 1$ for any $t \in I$, it follows directly from (2.23) that λ is parameterized by arc length. The converse also holds (cf. Exercise 2.4.1). For instance, the helix

$$\lambda(t) = \left(\cos\frac{t}{\sqrt{2}}, \sin\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$$

is a curve parametrized by arc length, since $\|\lambda'(t)\| = 1$ for every $t \in I$.

Proposition 2.4.1. Let $\lambda\colon I \to \mathbb{R}^3$ be a regular curve. Then, the arc length function (2.22) admits an inverse $h\colon J \to I$, defined on the open interval $J = s(I)$, and $\beta = \lambda \circ h$ is a reparametrization of λ that is parameterized by arc length.

Proof. Since λ is a regular curve, it follows that the arc length function s is strictly increasing, because

$$s'(t) = \|\lambda'(t)\| = 1,$$

for every $t \in I$. Thus, s admits an inverse $h\colon J \to I$. Since $h \circ s = id$, one has $\frac{dh}{ds} \cdot \frac{ds}{dt} = 1$, and therefore

$$\frac{dh}{ds} = \frac{1}{s'(t)} = \frac{1}{\|\lambda'(t)\|} > 0.$$

This implies that $\beta(s) = \lambda(h(s))$, $s \in J$, is a reparametrization of λ that satisfies

$$\|\beta'(s)\| = \left\|\frac{d\beta}{ds}\right\| = \left\|\frac{d\lambda}{dt} \cdot \frac{dh}{ds}\right\| = \frac{\|\lambda'(t)\|}{\|\lambda'(t)\|} = 1,$$

for every $s \in J$. Therefore, it follows from Exercise 2.4.1 that β is parameterized by arc length. $\square$

The curve β given by Proposition 2.4.1 is called a *reparametrization of* λ *by arc length.* Note that this reparametrization is not uniquely determined, because it depends of the arc length function, which depends of the instant t_0 fixed.

Example 2.4.2. Consider the helix parametrized by

$$\lambda(t) = (\cos t, \sin t, t),\ \ t \in \mathbb{R}.$$

Since $\|\lambda'(t)\| = \sqrt{2}$, the arc length function of λ, from $t_0 = 0$, is $s(t) = \sqrt{2}t$, and its inverse is given by $h(s) = \frac{1}{\sqrt{2}}s$. Thus,

$$\beta(s) = \lambda(h(s)) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$$

is a reparametrization of λ by arc length.

Let $\lambda\colon I \to \mathbb{R}^3$ be a curve parametrized by arc length s. Since $\|\lambda'(s)\| = 1$ for every $s \in I$, the norm $\|\lambda''(s)\|$ of the second derivative measures the rate of change of the tangent line to λ, that is, how rapidly the curve pulls away from the tangent line at s.

Definition 2.4.3. Let $\lambda\colon I \to \mathbb{R}^3$ be a curve parametrized by arc length s. The number $k(s) = \|\lambda''(s)\|$ is called the *curvature* of λ at s.

If λ is a straight line, parametrized by $\lambda(s) = as + b$, with $\|a\| = 1$, then $k(s) = 0$. Conversely, if $k(s) = 0$ for every s, then by integration one has $\lambda(s) = as + b$, and the curve is a straight line. Therefore, a curve parametrized by arc length has vanishing curvature if and only if its trace is contained in a straight line.

Example 2.4.4. Consider a helix parametrized by

$$\lambda(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}\right),\ s \in \mathbb{R},$$

where a, b, c are positive constants, with $a^2 + b^2 = c^2$. Note that the parameter s is the arc length, and a straightforward computation shows that $k(s) = \frac{a}{c^2}$, for every $s \in \mathbb{R}$.

At points where $k(s) \neq 0$, a unit vector $n(s)$ is well defined by the equation

$$\lambda''(s) = k(s) \cdot n(s).$$

Since $\lambda''(s)$ is orthogonal to $\lambda'(s)$, it follows that $n(s)$ is orthogonal to $\lambda'(s)$, and it is called the *normal vector* of λ at s.

Consider now a regular surface M in $\mathbb{R}^3$. Fixed a point $p \in M$, consider a regular curve $\gamma\colon (-\epsilon, \epsilon) \to M$, with $\gamma(0) = p$, and set $\cos\theta = \langle n(0), N(p)$ the angle between the normal vector n to γ and the normal vector N to M at p. Denoting by k the curvature of γ at p, the number

$$k_n = k\cos\theta$$

is called the *normal curvature* of the curve γ at $p \in M$. Note that k_n is the length of the projection of the vector kn over the normal N to the surface at the point p, with a sign given by the orientation of M at p.

In order to interpret the quadratic form II_p, defined in (2.20), let us consider a curve parametrized by arc length $\gamma\colon (-\epsilon, \epsilon) \to M$, with $\gamma(0) = p$. Denoting by $N(s) = N(\gamma(s))$, one has $\langle N(s), \gamma'(s)\rangle = 0$. Thus,

$$\langle N(s), \gamma''(s)\rangle = -\langle N'(s), \gamma'(s)\rangle.$$

Therefore,

$$\begin{aligned}\mathrm{II}_p(v) &= \langle A\gamma'(0), \gamma'(0)\rangle = -\langle dN(p)\cdot\gamma'(0), \gamma'(0)\rangle \\ &= -\langle N'(0), \gamma'(0)\rangle = \langle N(0), \gamma''(0)\rangle \\ &= \langle N, kn\rangle(p) = k_n(p).\end{aligned} \tag{2.24}$$

That is, the value of the quadratic form II_p in a unit vector $v \in T_pM$ is equal to the normal curvature of a curve parametrized by arc length through p and tangent to v. In particular,

Corollary 2.4.5 (Meusnier)**.** All curves lying on a regular surface M and having at a given point $p \in M$ the same tangent line have at this point the same normal curvatures.

Given an orientable regular surface M in $\mathbb{R}^3$ and fixed a point $p \in M$, consider the shape operator $A = -dN(p)$ of M at p and the corresponding orthonormal basis $\{e_1, e_2\}$ of T_pM that diagonalizes A as in (2.19). The numbers k_1 and k_2 ($k_1 \geq k_2$) are the maximum and minimum, respectively, of the quadratic form II_p, restricted to the unit circle of T_pM. In other words, according to (2.24), k_1 and k_2 are the extreme values of the normal curvatures at p. Such extreme values k_1 and k_2 are called the *principal curvatures* of M at p; the corresponding eigenvectors e_1, e_2 are called the *principal directions* at p.

The knowledge of the principal curvatures of M at p allows us to compute the normal curvature along any given unit vector $v \in T_pM$. In fact, write

$$v = \cos\theta \cdot e_1 + \sin\theta \cdot e_2,$$

where θ is the angle between e_1 and v in the orientation of T_pM. Thus, it follows from (2.24) that the normal curvature at p along v is given by

$$\begin{aligned} k_n(p) &= \mathrm{II}_p(v) = -\langle dN(p) \cdot v, v\rangle = \langle Av, v\rangle \\ &= \langle k_1 \cos\theta \cdot e_1 + k_2 \sin\theta \cdot e_2, \cos\theta \cdot e_1 + \sin\theta \cdot e_2\rangle \\ &= k_1 \cos^2\theta + k_2 \sin^2\theta. \end{aligned} \tag{2.25}$$

The expression (2.25) is known as the *Euler formula.*

This description, as well as the one that follows, was introduced by Euler. Given an orientable regular surface M and a point $p \in M$, fix an unit vector $v \in T_pM$. The plane π_v, determined by v and the unit vector $N(p)$ normal to M, intersects the surface M along a plane curve γ_v, called the *normal section* of M at p along v. Since the normal vector n is orthogonal to $\gamma_v'(0)$ and is contained in the plane π_v, it follows that n and N are parallel. Therefore, the curvature k_v of γ_v at p is equal to the absolute value of the normal curvature at p along v. Thus, the principal curvatures k_1 and k_2 are the maximum and minimum values of the curvatures k_v, with $\|v\| = 1$.

From Examples 2.3.2 and 2.3.3, the principal curvatures of a plane $\mathcal{P}$ and the unit sphere $\mathbb{S}^2$ are constant and equal to 0 and 1, respectively. From Example 2.3.4, it follows that the principal curvatures of the cylinder $\mathcal{C}$ are equal to 0 and 1 at all points. Finally, from Example 2.3.5, the principal curvatures of the hyperbolic paraboloid $\mathcal{H}$ at the point $(0, 0, 0)$ are equal to -2 and 2.

A point $p \in M$ is called an *umbilical point* of the surface M if the principal curvatures k_1 and k_2 are equal at p. For instance, all the points of a sphere and a plane are umbilical points. The following result tell us that the only surfaces which contain only umbilical points are spheres and planes.

Proposition 2.4.6. If all the points of a connected surface M are umbilic, then M is contained in a plane or a sphere.

Exercises

1. Show that a regular curve $\lambda\colon I \to \mathbb{R}^3$ is parameterized by arc length if and only if $\|\lambda'(t)\| = 1$ for every $t \in I$.

2. Assume that all normals of a parametrized curve cross through a fixed point. Prove that the trace of the curve is contained in a circle.

3. Let $\lambda\colon I \to \mathbb{R}^3$ be a regular curve such that all its tangent lines cross through a fixed point. Prove that the trace of λ is contained in a straight line.

4. Calculate the curvature of the *catenary*, which is the curve $\lambda\colon \mathbb{R} \to \mathbb{R}^2$ parametrized by $\lambda(t) = (t, \cosh t)$.

5. Show that if a regular surface is tangent to a plane along a curve, then at the points of this curve, at most one of the principal curvatures is non-zero.

6. Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in M$, is constant.

7. Describe the region of the unit sphere covered by the image of the Gauss map of the paraboloid of revolution $z = x^2 + y^2$.

Bibliography

[1] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

[2] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. 3, 3rd Edition, Publish or Perish, 1999.