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Chimera states through invariant manifold theory

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Abstract

We establish the existence of chimera states, simultaneously supporting synchronous and asynchronous dynamics, in a network of two symmetrically linked star subnetworks of identical oscillators with shear and Kuramoto–Sakaguchi coupling. We show that the chimera states may be metastable or asymptotically stable. If the intra-star coupling strength is of order ε, the chimera states persist on time scales at least of order $1/\varepsilon$ in general, and on time-scales at least of order $1/\varepsilon^2$ if the intra-star coupling is of Kuramoto–Sakaguchi type. If the intra-star coupling configuration is sparse, the chimeras are asymptotically stable. The analysis relies on a combination of dimensional reduction using a Möbius symmetry group and techniques from averaging theory and normal hyperbolicity.

Keywords: chimera states, complex networks, normal hyperbolicity, bifurcations
Mathematics Subject Classification numbers: 37D10, 70K50, 34C29.

S Supplementary material for this article is available online

(Some figures may appear in colour only in the online journal)

1. Introduction

In 2002, Kuramoto and Battogtokh [15] observed the coexistence of spatiotemporal synchronous and asynchronous oscillations in a ring of identical coupled oscillators. This

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phenomenon of partial synchronization in networks of identical coupled oscillators was subsequently branded *chimera* and observed in a wide range of experimental settings, including lasers [16], photoelectrochemical oscillators [9, 24] and coupled metronomes [17]. For comprehensive reviews of the prolific occurrence of chimeras in physical and numerical experiments, see [11, 21].

On the theoretical side, various mechanisms for the occurrence of chimeras have been proposed [1, 19, 21, 29]. In the limit of infinitely many oscillators, a spectral theory for chimera states was developed [20]. Notably, the understanding of chimera states in large but finite size networks is more challenging [28]. Carefully designed coupling functions have been shown to generate chimeras in finite size systems [3, 5]. Overall, despite a good two decades of fascination with chimeras, we remain far from a comprehensive mathematical understanding of their occurrence and behavior.

The main objective of this paper is to obtain mathematical results on chimera behavior in a large but finite size network of identical oscillators, with a specific modular structure and a relatively general type of coupling. The example concerns the union of two identical, symmetrically coupled star subnetworks. The key ingredients that underlie our analysis are a dimensional reduction due to a Möbius group symmetry [27], averaging theory [22] and normally hyperbolic invariant manifold theory [8, 10].

2. Model and discussion of main results

The network we consider is depicted in figure 1. It consists of two symmetrically coupled star subnetworks labelled '+' and '−'. Each star subnetwork consists of a central *hub* connected to *N leaves*. The isolated dynamics of all nodes are described by identical phase oscillators, and we denote hub states as $\varphi_0^{\pm} \in S^1$ and leaf states as $\varphi_i^{\pm} \in S^1$, $i = 1, \ldots, N$, where $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, with equations of motion

$$\begin{aligned} \dot{\varphi}_0^+ &= \omega + \lambda \sum_{j=1}^{N} H(\varphi_j^+, \varphi_0^+), \\ \dot{\varphi}_i^+ &= \omega + \lambda H(\varphi_0^+, \varphi_i^+) + \varepsilon\lambda c h(\varphi_i^- - \varphi_i^+), \end{aligned} \tag{1}$$

and likewise for the '−' star. Here, ω is the oscillator natural frequency and λ is the coupling strength between hubs and leaves. The function H is the Kuramoto–Sakaguchi coupling

$$H(\varphi_j, \varphi_i) = c + \sin(\varphi_j - \varphi_i + \delta), \tag{2}$$

where $c \neq 0$ is the shear and $\delta \in (0, \pi/4)$ is the phase frustration. This coupling is related to the phase reduction of an Andronov–Hopf bifurcation [14]. Note that shear parameter c causes the frequency of the oscillators to depend on node degree. Moreover, h is a diffuse pairwise coupling function between the two stars. The inter-coupling strength of the interaction between the star motif components is represented by ε. Our parameterization of the inter-coupling strength as $\varepsilon\lambda c$ leads to simpler mathematical statements.

We express the level of synchrony in terms of complex order parameters

$$z^+ = \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}(\varphi_j^+ - \varphi_0^+)} \quad \text{and} \quad z^- = \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}(\varphi_j^- - \varphi_0^-)}, \tag{3}$$

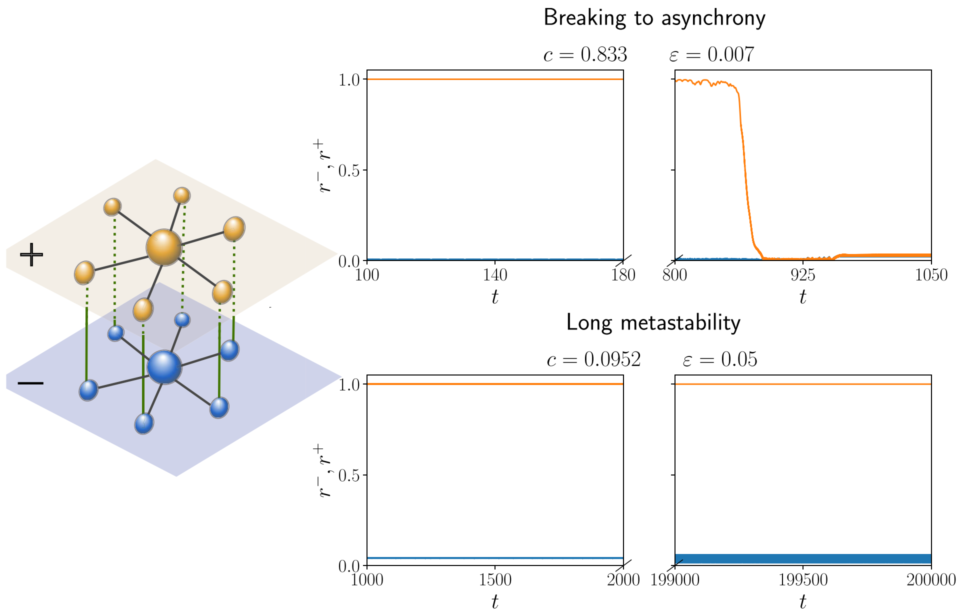

Figure 1. Metastability of chimera states in symmetrically coupled star networks. In the left part, two symmetrically coupled stars are depicted. In the right part, the time series of order parameters r^+ and r^- for star subnetworks $+$ and $-$ are presented. The upper right plot displays a chimera collapse to asynchrony. The lower right plot shows a chimera that does not break after 10^4 cycles. We consider $N = 200$, $h(\varphi_j^{\mp} - \varphi_i^{\pm}) = \sin(\varphi_j^{\mp} - \varphi_j^{\pm} + \delta)$, $\delta = 0.3$, $\omega = 1$ and $\lambda = 1$, where c is the shear parameter and ε the intra-star coupling strength.

related to the average phase difference between corresponding phase oscillators in the two-star motif subnetworks. We also consider

$$r^+ := |z^+| \quad \text{and} \quad r^- := |z^-|$$

the real order parameter of each subnetwork. Here, $r = 1$ refers to full synchrony and $r = 0$ represents full asynchrony when phases tend to be uniformly distributed.

Earlier observations of Ko and Ermentrout [12, 13], Vlasov *et al* [26] and Toenjes *et al* [23] based on numerics and $N \to \infty$ approximation suggest that phase oscillators in star networks exhibit coexisting synchronous and asynchronous states caused by shear. When $\varepsilon = 0$, each star exhibits coexistence of a stable state in which all leaves are in *synchrony* and another stable state in which the motion of the leaves is *asynchronous*.

We consider the dynamics of the coupled subnetworks ($\varepsilon > 0$) with shear parameter chosen in the bi-stability regime of the isolated dynamics and initial conditions for each subnetwork are chosen near the alternative stable states, so that the resulting dynamics in the absence of coupling ($\varepsilon = 0$) would display a combination of synchronous and asynchronous dynamics. In the presence of coupling ($\varepsilon > 0$), numerical experiments show that such states tend to persist (to the limit of our simulation time, approximately 10^6 cycles of natural frequency) when the coupling strength ε between the star motif is small, or collapse to a completely synchronized or asynchronous state when this coupling strength is not so small. Representative examples from our simulations are presented in figure 1.

In this paper, we address the explanation of these observations for large N and small ε. First, we prove that in the uncoupled limit ($\varepsilon = 0$), synchronous and asynchronous motion coexist. The synchronous motion corresponds to the existence of an invariant normally hyperbolic manifold M_+ and the asynchronous motion corresponds to the existence of an invariant manifold normally hyperbolic with boundary M_-. We prove (theorem 6.1) that the coupled stars system has an invariant normally hyperbolic manifold M_ε that is close to $M_+ \times M_-$. The chimera states persist for all time only if the trajectories of the system remain on M_ε. However, since M_ε has a boundary ∂M_ε the trajectories may drift along M_ε and escape which may lead to chimera breaking. We prove that existence of three classes of chimeras:

(*Short metastability*) Chimera states exist for a time $\mathcal{O}(\varepsilon^{-1})$,[3] see theorem 6.2 (a).

(*Long metastability*) If the intra-star coupling is of Kuramoto–Sakaguchi type then chimera states exist for a time $\mathcal{O}(\varepsilon^{-2})$, see theorem 6.2 (b).

(*Stable*) Chimera states, which persist for all time, exist if the intra-star coupling configuration is sufficiently sparse, see theorem 6.4.

Our main results rely on estimates from normally hyperbolic invariant manifold and averaging theory to obtain bounds on the drift on M_ε. Before discussing the proofs of the main results in more detail, in the next section we present numerical observations that have motivated the theory developed in this paper.

3. Change of coordinates and parameterization

We first bring the equations to a suitable form for simulations and theory. To this end, we perform the variable substitution $\varphi_j \mapsto \tilde{\varphi}_j := \varphi_j + \omega t$ for all $j = 0, \ldots, N$ and a time rescaling $t \mapsto \tau := \lambda c t$. This rescales equation (1) and turns the degree into the hub node's natural frequency

$$\begin{aligned} \dot{\varphi}_0^+ &= N + \frac{1}{c}\sum_{j=1}^{N} \sin(\varphi_j^+ - \varphi_0^+ + \delta), \\ \dot{\varphi}_i^+ &= 1 + \frac{1}{c}\sin(\varphi_0^+ - \varphi_i^+ + \delta) + \varepsilon h(\varphi_i^- - \varphi_i^+), \quad i = 1, \ldots, N, \end{aligned} \tag{4}$$

where we abuse notation by re-using variables φ and denoting differentiation with respect to τ by $\cdot$ again. Let us denote $\sigma = 1/c$. We also allow for a slight generalization, where $\beta > 1$ is the hub frequency. When $\beta = N$, we recover the previous equation (4). Here β works as a frequency gap between hubs and leaves. So, for the $+$ and $-$ subnetworks the equations of motion are

$$\begin{aligned} \dot{\varphi}_0^+ &= \beta + \beta\frac{\sigma}{N}\sum_{j=1}^{N} \sin(\varphi_j^+ - \varphi_0^+ + \delta) \\ \dot{\varphi}_i^+ &= 1 + \sigma \sin(\varphi_0^+ - \varphi_i^+ + \delta) + \varepsilon h(\varphi_i^- - \varphi_i^+), \quad i = 1, \ldots, N, \end{aligned} \tag{5}$$

and the ones for the $-$ star are similar with $+$ and $-$ sign swapped.

[3] Here $\mathcal{O}(\cdot)$ stands for Landau's symbols for order functions. So, $t = \mathcal{O}(\delta(\varepsilon))$ is: there exists $\varepsilon_0 \geqslant 0$ and $K \geqslant 0$ such that $0 \leqslant t \leqslant K|\delta(\varepsilon)|$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$.

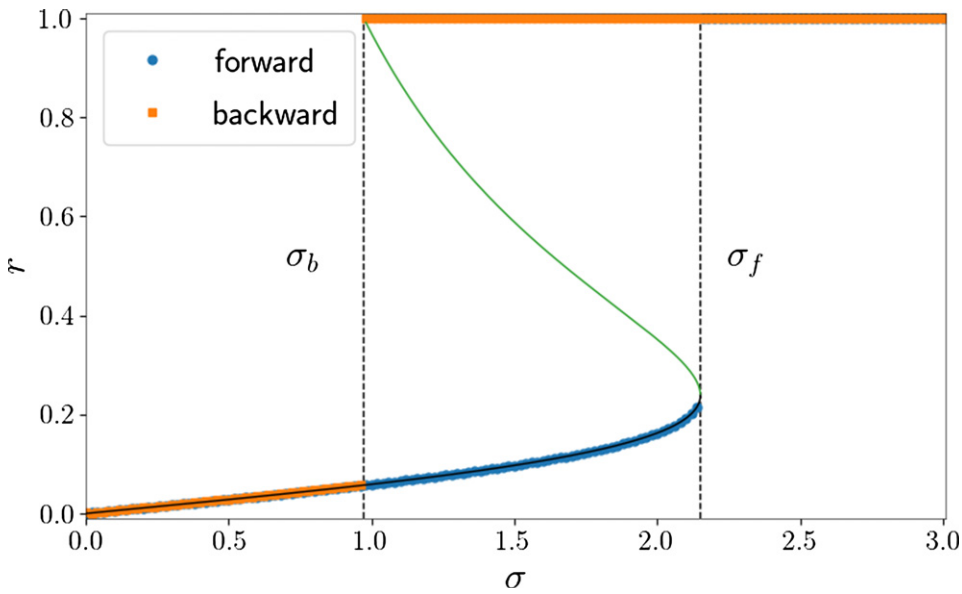

Figure 2. Synchronization diagram for a single star with $N = 200$ leaves. The parameters are $\beta = 10$, $\delta = 0.3$ and at each step the coupling strength was increased (decreased) by $\Delta\sigma = 0.02$. Dots are the values of the order parameter calculated numerically. Solid lines are the theoretical predictions and the green solid line is the separatrix. The region between the two vertical dashed lines is where both synchronous and asynchronous states coexist.

4. Numerical observations

We begin by establishing the synchronisation diagram for a single star and then we proceed to numerically explore chimera states in networks.

4.1. Collective dynamics of a single star network

We obtain the bifurcation diagram of the order parameter for a single star, see figure 2.

4.1.1. Numerical procedure. We integrate equations (5) of one star using an implicit Runge–Kutta method of order five (we use a solver described in [2]). We start at $\sigma = 0$ with uniformly randomly distributed initial phases in $(0, 2\pi)$ and evaluate the order parameter r in the stationary regime. Then we increase adiabatically the coupling by $\Delta\sigma = 0.02$ and, using the outcome of the last run as the initial condition, calculate the new value of the stationary order parameter r at $\sigma + \Delta\sigma$, repeating these steps until a maximal value σ is reached. This curve is called the forward continuation. The order parameter increases slowly and smoothly until $\sigma = \sigma_{\mathrm{f}}$, where it jumps discontinuously to $r = 1$.

We obtain the backward continuation by decreasing adiabatically the coupling strength by steps of size $\Delta\sigma$ from the synchronous state. We use the outcome of the last run and add a small random uniform number drawn from the interval $(0, 0.01)$ as the initial condition. In the backward continuation, the order parameter drops abruptly to the asynchronous branch.

4.1.2. Main findings. In section 5, we prove the existence of this synchronization diagram via the theory of invariant manifolds. In figure 2, the region between the dashed lines corresponds

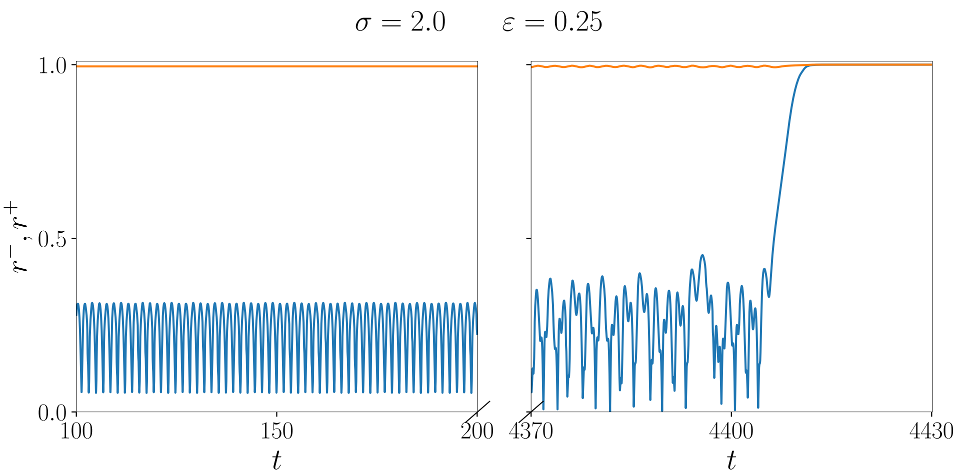

Figure 3. Chimera breaking to synchrony. We show the time series of order parameters r^+ and r^- for the star subnetworks depicted in figure 1. After approximately 4.4×10^3 cycles of natural frequency the chimera collapses to synchrony. Parameter values are $N = 200$, $\beta = 10$, $\sigma = 2.0$ and $\varepsilon = 0.25$.

to the values of the coupling critical couplings

$$\sigma_\mathrm{b} = \frac{\beta - 1}{1 + \beta\,\cos(2\delta)} \quad \text{and} \quad \sigma_\mathrm{f} = \frac{\beta - 1}{\sqrt{1 + 2\beta\,\cos(2\delta)}}.$$

The backward critical point σ_b is calculated in section 5.3, where we prove that it is a consequence of requiring the existence and stability conditions for the synchronous state. We prove in theorem 5.8 that, for a large set of initial condition, when $\sigma < \sigma_\mathrm{b}$ even when we initialize the network in a neighborhood of the synchronization manifold the network will decay to an asynchronous motion. In other words, the backward transition is discontinuous. The forward critical value σ_f is calculated in section 5.4.

4.2. Symmetrically coupled stars

In figure 1, we presented examples of long metastable chimeras and chimeras breaking to asynchrony. Here, we illustrate chimeras breaking to synchrony. The stars are coupled with inter-coupling strength ε and coupling function

$$h(\varphi_j^{\mp} - \varphi_i^{\pm}) = \sin(\varphi_j^{\mp} - \varphi_i^{\pm} + \delta), \quad i, j \in \{1, \dots N\}.$$

Figure 3 shows that for $N = 200$, $\beta = 10$, $\sigma = 2.0$, and $\varepsilon = 0.25$, after 4400 cycles the chimera breaks to synchrony. We note that the oscillatory behavior in the numerical experiments is akin to the *breathing chimera* [1].

4.2.1. Chimera lifetime. The coupling strength between the stars provides the time scale the chimera exists—either of order $1/\varepsilon$ (in the general inter-coupling function) or $1/\varepsilon^2$ (for sinusoidal). We performed numerical experiments to check these predictions. We introduce a parameter ζ which splits the absolute value of the order parameter in theorem 6.1 measured on the synchronous and the asynchronous star. The chimera lifetime τ is the minimum time it

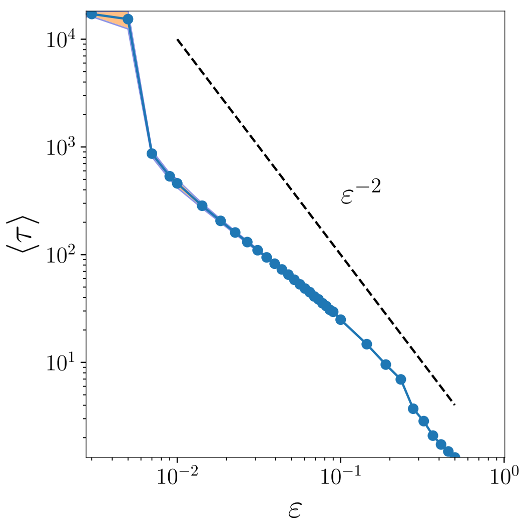

Figure 4. Chimera lifetime $\langle\tau\rangle$ as function of inter-coupling strength ε. Lifetime $\langle\tau\rangle$, represented by dotted lines, is an average over initial conditions chosen nearby M, while the shaded area is the standard deviation—for each ε in the interval $[0.001, 0.01]$ we pick five initial conditions, while when ε is inside the interval $[0.01, 0.5]$ we pick 50 initial conditions. The dashed line represent the scaling ε^{-2}. The parameters are $\beta = 200$, $\sigma = 1.2$ and $\zeta = 0.25$.

takes to violate the splitting condition from theorem 6.1, so when any of the next two conditions is satisfied:

$$|z^-(\tau) - z^-(0)| > \zeta \quad \text{or} \quad r^+(\tau) < 1 - \zeta.$$

The lifetime depends on the initial conditions and parameters of the network. We select initial conditions starting in a neighborhood of M.[4] And for each value of ε, we compute τ for different initial conditions and then compute the average lifetime $\langle\tau\rangle$. Figure 4 shows $\langle\tau\rangle$ as a function of ε which appears to agree with our prediction for moderate values of ε.

4.3. Coupled scale-free networks

Scale-free networks display power-law degree distribution $P(k) \propto k^{-\gamma}$, where k denotes the degree and P the corresponding number of nodes with that given degree k. Hubs in such scale-free networks are dominant and for large values of γ, star motifs are building blocks for scale-free networks [4]. Although our results for star graphs do not apply to the scale-free network, we can use them as heuristics to build chimera-like states in scale-free networks. Consider the following equations of motion on complex networks,

$$\dot{\varphi}_i = k_i + \sigma \sum_{j=1}^{N} A_{ij} \sin\left(\varphi_j - \varphi_i + \delta\right), \quad i = 1, \ldots, N, \tag{6}$$

[4] We select initial conditions satisfying the restrictions for system be in M and we add a small random uniform vector with each coordinate drawn from the interval $(0, 0.01)$.

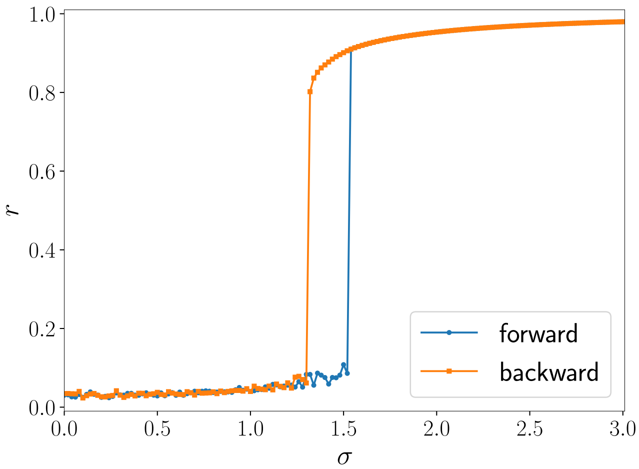

Figure 5. Synchronization diagram for a Barabási–Albert network. We fix $N = 1000$ and $\langle k\rangle = 6$. For $\delta = 0.03$, at each step we increase the coupling strength (and decreased respectively) by $\Delta\sigma = 0.02$. We observe for $\sigma \in [1.3, 1.5]$ is a region of coexistence of synchronous and asynchronous states.

where σ is the coupling strength, k_i is the degree of the vertex i and A_{ij} is the adjacency matrix. A_{ij} is equal to 1 if $i = j$ and 0 otherwise. The synchronization diagram for a Barabási–Albert [4] network with $N = 1000$ and mean degree $\langle k\rangle = 6$ is shown in figure 5.[5]

We generate two identical Barabási–Albert networks. The kth vertex of one network is coupled to (and only to) to the kth vertex of the other network. The coupling function used is the Kuramoto–Sakaguchi coupling. Again σ denotes the intra-coupling strength and the ε inter-coupling. We chose initial conditions inside the hysteresis loop in figure 5, starting one network in the synchronous and the other in the asynchronous branch. Figure 6 shows chimera-like states in the two coupled Barabási–Albert networks.

4.3.1. Chimera lifetime. We repeat the same procedure as for the coupled stars to quantify the chimera lifetime to the two identical coupled BA networks case. For a fixed initial condition, we compute the chimera lifetime τ for different values of ε. Figure 7 displays a qualitative scaling of the chimera lifetime with respect to the inter-coupling strength ε.

5. Mathematical analysis

In the remainder of this manuscript we prove our main results theorems 6.1–6.4. To this end we present the strategy to prove the main theorems.

5.1. Strategy

First step: *constructing the synchronous manifold of a single star*. There exists an invariant manifold where all phases are locked. In section 5.3 we prove that for certain intervals of the parameters the manifold is normally attracting, see proposition 5.4.

[5] For all simulations involving the BA networks, we integrate the equations using an explicit Runge–Kutta method of order four.

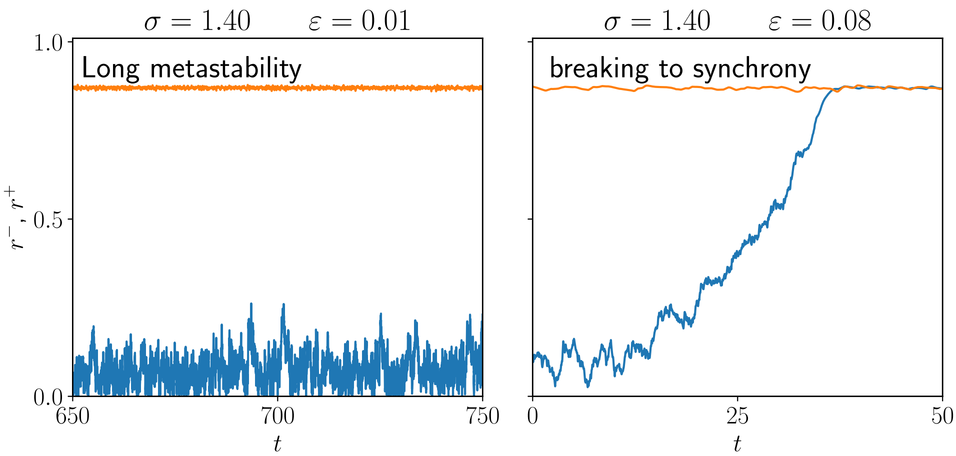

Figure 6. Chimeras in two identical coupled Barabási–Albert networks. Order parameters r^+ and r^- for two networks with $N = 1000$ and $\langle k \rangle = 6$. Left panel shows a metastable chimera. In right panel, the chimera quickly breaks to synchrony. We indicate the intra-coupling strength σ and inter-coupling strength ε.

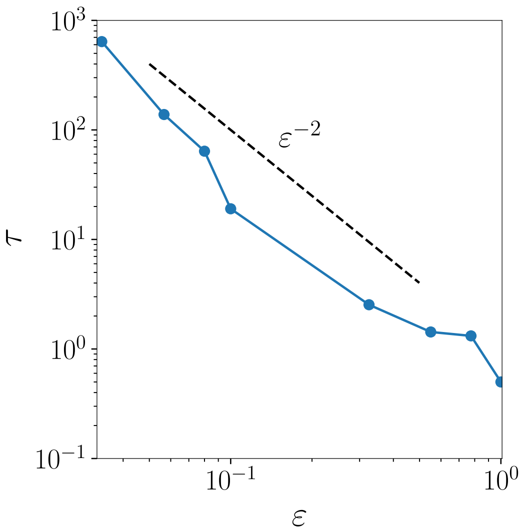

Figure 7. Chimera lifetime τ for two identical coupled BA networks. Dots correspond to chimera lifetime τ with respect to the inter-coupling ε for a fixed initial condition chosen near uniformly distributed phases for the asynchronous network and near synchronization for the other network. Dashed lines represent the scaling ε^{-2}. The parameters are $N = 1000$, $\langle k \rangle = 6$, $\sigma = 1.4$ and $\zeta = 0.35$.

Second step: *constructing the asynchronous manifold of a single star.*

(a) Reduction in terms of Möbius actions: we rewrite the model in terms of relative phases. In these coordinates, the dynamics is described in a low-dimensional manner by the action of a three-parameter Möbius group, see section 5.4.1. The Watanabe–Strogatz approach provides a differential equation for the parameters of the Möbius group $\alpha \in \mathbb{C}$ and $\psi \in S^1$ keeping invariant a set of constants of motion.

(b) Construct open sets of initial conditions such that the order parameter z is C^k close to α, see section 5.4.2 in lemma 5.6. The differential equation for α depends on z and by a small perturbation we obtain an equation solely in terms of α, see section 5.4.3 in theorem 5.7. By averaging theory, in section 5.4.4, we prove that the perturbed dynamics of α enters in finite time a neighborhood of zero, see theorem 5.8. This determines that the bifurcation diagram of a single star network is in fact discontinuous for sufficiently small coupling strength.

(c) By persistence of hyperbolic manifolds we obtain the invariant manifold corresponding to the asynchronous dynamics in the full equations, see proposition 5.9.

Third step: *constructing chimeras in coupled stars*. We obtain the existence of chimera states in the couple stars, see theorem 6.1, as a consequence of persistence of a product manifold for $\varepsilon = 0$. We show three distinct results about the time scale these chimera states exist depending on how generic we couple both stars.

Short metastability. We estimate an upper bound for how long it takes the perturbed dynamics of the constants of motion drift out the invariant manifold.

Long metastability. We assume the coupling function between both stars has sinusoidal form as an extra assumption. This implies the coupling term can be split up into two terms and restricted to the invariant manifold. This restriction introduces an effective coupling term of order ε^2 to the constants of motion dynamics.

Stability of sparse one-to-one coupling. We show that coupling a single leaf may generate a drift of the conserved quantities, but this will happen only for a single constant. That is, the drift does not cascade to other leaves. Hence, if only a small fraction of leaves are coupled this drift is immaterial and the order parameter will stay in a neighborhood of α and the chimera state is stable, see theorem 6.4.

Those estimates are valid while the coupled stars remain in the perturbed product manifold with boundary. If the constants of motion of the Watanabe–Strogatz approach drift out of the boundary of the asynchronous manifold, we lose control over the dynamics because the order parameter may no longer be in a neighborhood of α.

5.2. Preliminaries on invariant manifolds

We briefly recall some established results from the theory of normally hyperbolic invariant manifolds [7, 8, 10]. Such manifolds can be viewed as generalizations of hyperbolic fixed points and we use their property that they persist under small perturbations of the dynamics. We consider normally *attracting* invariant manifolds (NAIMs), i.e. those which only have stable normal directions, not unstable ones.

Let $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ define a dynamical system; we use $\mathbb{R}^n$ for simplicity, but it can be replaced by a smooth manifold. A submanifold $M \subset \mathbb{R}^n$ is said to be a NAIM for f if it is invariant under the flow Φ^t of f and furthermore the linearized flow along M contracts normal directions, and does so more strongly than it contracts directions tangent to M. Our precise definition below corresponds to that of 'eventual absolute normal hyperbolicity' according to [10].

Definition 5.1 (Normally attracting invariant manifold). Let $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f \in C^1$ describe a dynamical system with flow $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Then a submanifold $M \subset \mathbb{R}^n$ is called a NAIM of f if all the following conditions hold true:

(a) M is invariant, i.e. $\forall\ t \in \mathbb{R} : \Phi^t(M) = M$;

(b) There exists a continuous splitting

$$T_M\mathbb{R}^n = TM \oplus N \tag{7}$$

of the tangent bundle $T\mathbb{R}^n$ over M into the tangent and a (stable) normal bundle, with continuous projections π_M, π_N and this splitting is invariant under the tangent flow $T\Phi^t = T\Phi^t_M \oplus T\Phi^t_N$;

(c) There exist real numbers $a < b \leqslant 0$ and $C > 0$ such that the following exponential growth conditions hold on the subbundles:

$$\forall\ t \leqslant 0,\ (m, v) \in TM : \quad \|T\Phi^t_M(m)\, v\| \leqslant C\,\mathrm{e}^{b\,t}\,\|v\|,$$

$$\forall\ t \geqslant 0,\ (m, v) \in N : \quad \|T\Phi^t_N(m)\, v\| \leqslant C\,\mathrm{e}^{a\,t}\,\|v\|.$$

Note that if we have two dynamical systems f_1, f_2 which each have an NAIM M_1, M_2 respectively and $\max(a_1, a_2) < \min(b_1, b_2)$, then $M_1 \times M_2$ is an NAIM for the product system. In our case, the dynamics on the invariant manifolds will be (close to) neutral, so $b_{1,2} \approx 0$ and hence the product system has an NAIM too.

A main result on normal hyperbolicity is that NAIMs persist under small perturbations of the vector field f. That is, we have the following theorem, see [8, theorem 1] or [10, theorem 4.1].

Theorem 5.2 (Persistence of NAIMs). *Let $M \subset \mathbb{R}^n$ be a compact NAIM for f. Then there exists an $\varepsilon > 0$ such that for any vector field $\tilde{f}$ with $\|\tilde{f} - f\|_{C^1} \leqslant \varepsilon$, there exists a unique manifold $\tilde{M}$ that is diffeomorphic and $\mathcal{O}\left(\|\tilde{f} - f\|_{C^1}\right)$-close to M and invariant under $\tilde{f}$. Furthermore, $\tilde{M}$ is an NAIM for $\tilde{f}$.*

The closeness of $\tilde{M}$ to M can in general be expressed as follows: $\tilde{M}$ can be described by the graph of a section of the normal bundle of M, and this section is C^1-small. In our case M can explicitly be given by the graph of a function, hence $\tilde{M}$ will be given by the graph of a C^1-small perturbation of this function.

In our case the NAIM M will have a boundary. Theorem 5.2 still holds true as long as the vector field f is pointing strictly outward at the boundary ∂M (also called 'overflowing invariant'), see [8]. In our case M will not be overflowing invariant. This can be overcome by artificially modifying f near the boundary ∂M. A slightly simpler approach uses [7, theorem 4.8]: we modify f such that it preserves a manifold S that transversely intersects M with $S \cap M = \partial M$. For example, S could be the local normal bundle of M restricted to ∂M.

5.3. An NAIM for the synchronous state

The synchronous state is characterized by $r = 1$ and the phase locking manifold

$$M_\phi = \{\phi_1 = \cdots = \phi_N = \phi \in S^1, \phi - \varphi_0 = \Delta\phi \in S^1\} \subset \mathbb{T}^{N+1}$$

satisfies this condition on the order parameter. The stability of M_ϕ depends on the frequency mismatch, the phase frustration δ and intra-coupling strength σ [25]. The following hypothesis is necessary to prove existence of the synchronous manifold in proposition 5.4.

Hypothesis 5.3. Assume that

$$\sigma > \sigma_\mathrm{b} := \frac{\beta - 1}{1 + \beta\,\cos(2\delta)} \quad \text{and} \quad \arg(v) - \sin^{-1}\left(\frac{1-\beta}{\sigma\|v\|}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{8}$$

where $v = (\beta\ \sin(2\delta), 1 + \beta\ \cos(2\delta)) \in \mathbb{R}^2$, and hence $\|v\| = \sqrt{1 + \beta^2 + 2\beta\ \cos(2\delta)}$. Here $\arg(v)$ is the angle of v, as under the identification $\mathbb{R}^2 \cong \mathbb{C}$.

Proposition 5.4. *Let $\sigma > 0$, $\beta > 1$ and $\delta \in (0, \frac{\pi}{4})$. The synchronized state*

$$M_C = \{\ \phi_1 = \cdots = \phi_N = \phi^C \in S^1\ \} \tag{9}$$

is a NAIM for the dynamics (14) if and only if hypothesis 5.3 holds and when ϕ^C is given by

$$\phi^C = \delta - \pi + \arctan\left(\frac{1 + \beta\ \cos(2\delta)}{\beta\ \sin(2\delta)}\right) + \arccos\left(\frac{\beta - 1}{\sigma\|v\|}\right). \tag{10}$$

Proof. First we search for the value of $\phi = \phi^C$ such that M_C is an invariant manifold. Evaluating (14) on M_C we see that this amounts to

$$\dot{\phi} = 1 - \beta - \sigma\ \sin(\phi^C - \delta) - \beta\sigma\ \sin(\phi^C + \delta) = 0.$$

This can be rewritten as

$$\langle v\left(\cos(\phi^C - \delta), \sin(\phi^C - \delta)\right)\rangle = -\frac{\beta - 1}{\sigma} \tag{11}$$

with $v = (\beta \sin(2\delta), 1 + \beta \cos(2\delta))$. Hence we have solutions if

$$\|v\| = \sqrt{1 + \beta^2 + 2\beta\ \cos(2\delta)} \geqslant \frac{\beta - 1}{\sigma}. \tag{12}$$

To evaluate stability of M_C, let $(\dot{\phi}_1, \ldots, \dot{\phi}_N, \dot{\varphi}_0) = F(\phi_1, \ldots, \phi_N, \varphi_0)$ denote the system (14). Then we have the total derivative

$$DF|_{M_C} = -\sigma\ \cos(\phi^C - \delta)\begin{pmatrix} 1 & & & \emptyset \\ & \ddots & & \\ & & 1 & \\ \emptyset & & & 0 \end{pmatrix}$$
$$- \sigma\frac{\beta}{N}\ \cos(\phi^C + \delta)\begin{pmatrix} 1 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & 0 \\ -1 & \cdots & -1 & 0 \end{pmatrix}.$$

Thus, we find that $(0, \ldots, 0, 1)$ is an eigenvector with eigenvalue 0, that is, the direction along M_C is neutral. In the directions transversal to M_C we have the eigenvector

$$v_1 = \left(1, \ldots, 1, -\frac{\beta\ \cos(\phi^C + \delta)}{\cos(\phi^C - \delta) + \beta\ \cos(\phi^C + \delta)}\right)$$

with eigenvalue $\lambda_1 = -\sigma\left[\cos(\phi^C - \delta) + \beta\ \cos(\phi^C + \delta)\right]$ and $N - 1$ independent eigenvectors $v_i = (0, \ldots, 1, -1, 0, \ldots, 0)$ with eigenvalue $\lambda_2 = -\sigma \cos(\phi^C - \delta)$. So M_C is normally attracting if $\sigma > 0$ and

$$\cos(\phi^C - \delta) > 0 \quad \text{and} \quad \cos(\phi^C - \delta) + \beta\ \cos(\phi^C + \delta) > 0.$$

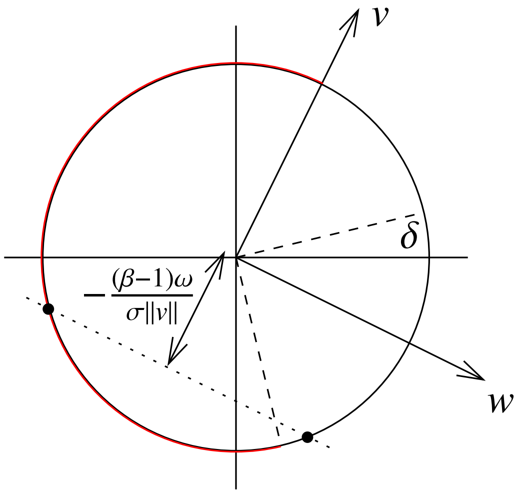

Figure 8. Geometric representation of the existence and stability of the synchronized state. Dots indicate the fixed points for $\phi^C \in S^1$ and the red arc is the unstable region where at least one condition in (13) is violated. The vectors v and w and the dashed lines are perpendicular.

These conditions can be rewritten as

$$\cos(\phi^C - \delta) > 0 \qquad \text{and} \qquad \langle w \left(\cos(\phi^C - \delta), \sin(\phi^C - \delta)\right)\rangle > 0, \tag{13}$$

with $w = (1 + \beta \cos(2\delta), -\beta \sin(2\delta))$ equal to v rotated clockwise over 90 degrees. Interpreting the conditions (11) and (13) geometrically with $\phi^C \in S^1$, see figure 8, we see that there are at most two synchronized states. Since $v \perp w$, precisely one of these states satisfies the second stability condition of (13) if and only if (12) holds as a strict inequality. Then that state M_C is actually stable if $\cos(\phi^C - \delta) > 0$ also holds, which can be expressed as the second condition in (8) and solving for ϕ^C yields (10).

The backward critical coupling σ_{b} is obtained by solving the equation $\cos(\phi^C - \delta) = 0$ and using the expression obtained for ϕ^C. This yields the first condition in (8). Thus both conditions in hypothesis 5.3 are necessary and by construction also sufficient to guarantee that M_C is an NAIM. □

5.4. An NAIM for the asynchronous state

Consider a single star network in system (5). We change to a new set of coordinates defined by $\phi_k = \varphi_k - \varphi_0$, measuring the phase difference between the leaves and the hub, and the resulting equations are

$$\begin{aligned} \dot{\phi}_i &= 1 - \beta - \sigma\ \sin(\phi_i - \delta) - \beta\frac{\sigma}{N}\sum_{j=1}^{N}\ \sin(\phi_j + \delta), \quad i = 1, \ldots, N, \\ \dot{\varphi}_0 &= \beta + \beta\frac{\sigma}{N}\sum_{j=1}^{N}\ \sin(\phi_j + \delta). \end{aligned} \tag{14}$$

In the original coordinates the system had a global S^1-symmetry by which we have effectively reduced the system, since the equations for the relative angles ϕ_i are now decoupled from φ_0.

To find a stable asynchronous state, we change the equations for a single star from relative phase form (14) to new variables introduced by Watanabe–Strogatz [27].

5.4.1. Low-dimension description. Watanabe and Strogatz [27] showed that a class of systems of N identical coupled oscillators can be described by a set of three ODEs. They achieved this through a change of variables, that explicitly shows that the system possesses $N-3$ constants of motion. In [18] Marvel, Strogatz and Mirollo provided an explanation in terms of Möbius actions. We follow [18] and consider systems of N identical coupled phase oscillators with equations of motion that can be put into the form

$$\dot{\phi}_j = f\,\mathrm{e}^{\mathrm{i}\phi_j} + g + \bar{f}\,\mathrm{e}^{-\mathrm{i}\phi_j}. \tag{15}$$

Here f is a smooth complex-valued function of the phases $\phi_1, \phi_2, \ldots, \phi_N \in S^1$, $\bar{f}$ is its complex conjugate, and g is a real-valued function of the phases $\phi_1, \phi_2, \ldots, \phi_N \in S^1$. The phases ϕ_j evolve in time according to the action of the Möbius group G,[6]

$$\mathrm{e}^{\mathrm{i}\phi_j(t)} = G_{\alpha(t),\psi(t)}(\mathrm{e}^{\mathrm{i}\theta_j}), \tag{16}$$

where θ_j are a set of constant (time independent) angles, and we use the group parameterization $G_{\alpha,\psi}$ and its action on $w \in \mathbb{C}$ given by

$$G_{\alpha,\psi}(w) = \frac{\alpha + \mathrm{e}^{\mathrm{i}\psi}\,w}{1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi}w}, \tag{17}$$

with $\psi \in S^1$ and $\alpha \in \mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$.

The time evolution of the phases in (16) satisfies the equations (15) as long as $\alpha(t)$ and $\psi(t)$ are solutions of the differential equations

$$\begin{aligned} \dot{\alpha} &= i\left(f\alpha^2 + g\alpha + \bar{f}\right), \\ \dot{\psi} &= f\alpha + g + \bar{f}\bar{\alpha}. \end{aligned} \tag{18}$$

The functions f and g still implicitly depend on $\phi_1, \phi_2, \ldots, \phi_N \in S^1$, so we have to use (16) to express the ϕ_j's in terms of α, ψ, and the θ_j's and obtain a closed system of equations for α, ψ. For suitable models, we can express f and g in terms of the order parameter z, i.e. $f = f(z, \bar{z})$ and $g = g(z, \bar{z})$, where z is given by

$$z(\alpha, \psi, \theta) := \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\phi_j} = \frac{1}{N}\sum_{j=1}^{N} G_{\alpha,\psi}\left(\mathrm{e}^{\mathrm{i}\theta_j}\right). \tag{19}$$

We now reformulate the Watanabe–Strogatz change of variables in a more formal geometric setting. Let $\dot{\phi} = X(\phi)$ with $X \in \mathfrak{X}(\mathbb{T}^N)$ denote the vector field of (15) and let $(\dot{\alpha}, \dot{\psi}, \dot{\theta}) = \hat{X}(\alpha, \psi, \theta)$ with $\hat{X} \in \mathfrak{X}(Q)$ denote the vector field associated to the Watanabe–Strogatz coordinates $(\alpha, \psi, \theta) \in Q := \mathbb{D} \times S^1 \times \mathbb{T}^N$, i.e. we have $\dot{\theta} = 0$ and $\dot{\alpha}, \dot{\psi}$ given by (18). Let Φ^t and $\hat{\Phi}^t$

[6] The complete Möbius group consists of all fractional linear transformations $G(w) = \frac{aw+b}{cw+d}$ of the complex plane such that $ad - bc \neq 0$ but we only consider the subgroup that preserves the unit circle. As in related literature we refer to this subgroup as the Möbius group.

denote the associated flows. Then we have the commuting diagram

$$\begin{tikzcd} Q \arrow[r, "\hat{\Phi}^t"] \arrow[d, "\pi"] & Q \arrow[d, "\pi"] \\ \mathbb{T}^N \arrow[r, "\Phi^t"] & \mathbb{T}^N \end{tikzcd} \tag{20}$$

where $\pi : Q \to \mathbb{T}^N$ is the submersion $\phi = \pi(\alpha, \psi, \theta)$ defined by (16). Since $\pi(\alpha, \psi, \,\cdot\,) : \mathbb{T}^N \to \mathbb{T}^N$ is given by the diagonal action of $G_{\alpha,\psi}$, it is a diffeomorphism and hence π indeed a submersion. Note that although we loosely speak of π as a coordinate transformation, strictly speaking it is not, since it is not injective. The diagram (20) implies that $T\pi \circ \hat{X} = X \circ \pi$ and hence $X = T\pi \circ \hat{X} \circ \pi^{-1}$ for any right-inverse π^{-1}. On the other hand, a vector field $Y \in \mathfrak{X}(\mathbb{T}^N)$ can be lifted to a vector field $\hat{Y} = R \circ X \circ \pi \in \mathfrak{X}(Q)$ satisfying (20), given a choice of a right-inverse R of $T\pi$. We now choose

$$R(\alpha, \psi, \theta) = T_\theta \left[\pi(\alpha, \psi, \,\cdot\,)^{-1}\right] : T_\phi \mathbb{T}^N \to T_\theta \mathbb{T}^N \subset T_{(\alpha,\psi,\theta)} Q \tag{21}$$

with $\phi = \pi(\alpha, \psi, \theta)$. Let $G'_{\alpha,\psi}$ denote the derivative of the action on S^1, then we have

$$R(\alpha, \psi, \theta) = \operatorname{diag}\left(G'_{\alpha,\psi}(\theta_1)^{-1}, \ldots, G'_{\alpha,\psi}(\theta_N)^{-1}\right). \tag{22}$$

The upshot is that if we perturb the original vector field X to $X + \varepsilon Y$, for example when introducing a coupling between the stars, then we can lift Y to $\hat{Y} \in \mathfrak{X}(Q)$ such that $\hat{Y}$ is given by

$$\dot{\alpha} = 0, \;\; \dot{\psi} = 0, \qquad \dot{\theta}_j = G'_{\alpha,\psi}(\theta_j)^{-1} Y_j(\pi(\alpha, \psi, \theta)). \tag{23}$$

In the case that only some of the components Y_j are nonzero, it follows that only the associated θ_j's are not conserved anymore by $\hat{Y}$, while the other θ_j's are still conserved.

On the other hand, we still lift the original vector field X using the Watanabe–Strogatz approach, such that its lift $\hat{X} \in \mathfrak{X}(Q)$ is given by (18) together with $\dot{\theta} = 0$.

5.4.2. Approximation of the complex order parameter. We write the sine functions in (14) in exponential form. This shows that we can recast the equations in the Watanabe–Strogatz form (15) with f and g given by

$$\begin{aligned} f(z, \bar{z}) &= \frac{-\sigma\, \mathrm{e}^{-\mathrm{i}\delta}}{2i}, \\ g(z, \bar{z}) &= 1 - \beta - \sigma\beta \left(\frac{z\, \mathrm{e}^{\mathrm{i}\delta} - \bar{z}\, \mathrm{e}^{-\mathrm{i}\delta}}{2i}\right). \end{aligned} \tag{24}$$

For a more detailed analysis of a stable asynchronous state in the system, we use that $z \approx \alpha$ when the θ's are close to being uniformly distributed on S^1 (called 'splay states'), when the number of leaves N is sufficiently large. Let us make this more precise.

Definition 5.5. We say that $\theta \in \mathbb{T}^N$ is uniformly distributed if there exists a $\varphi \in S^1$ such that $\theta_j = \frac{2\pi j}{N} + \varphi$ up to permutation. We denote the set of all such θ by Θ.

Lemma 5.6. *Fix $0 < r < 1$ and consider the disc $\mathbb{D}_r = \{\alpha \in \mathbb{C} | |\alpha| < r\}$. For each $k \geqslant 1$ and $\varepsilon > 0$ there exists for all N sufficiently large an open neighborhood U of $\Theta \subset \mathbb{T}^N$ such* *that the map* $\mathbb{D}_r \to \mathbb{C},\ \alpha \mapsto z(\alpha,\psi,\theta) - \alpha$, *with* $z(\alpha,\psi,\theta)$ *given by* (19), *is smaller than* ε *in* C^k*-norm, uniformly for all* $(\psi,\theta) \in S^1 \times U$.

Proof. Consider an element $\theta \in \Theta$ and wlog. Assume that it is of the form as in definition 5.5 without permutation of the indices j and that these run from 0 to $N-1$.

Below, we use the geometric series expansion $(1+x)^{-1} = \sum_{l\geqslant 0}(-x)^l$ with $x = \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi}\,\mathrm{e}^{\mathrm{i}\theta_j}$, which converges for any $|\alpha| < 1$. We find

$$
\begin{aligned}
z(\alpha,\psi,\theta) &= \frac{1}{N}\sum_{j=0}^{N-1}\frac{\alpha + \mathrm{e}^{\mathrm{i}\psi}\,\mathrm{e}^{\mathrm{i}\theta_j}}{1+\bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi}\,\mathrm{e}^{\mathrm{i}\theta_j}}\\
&= \frac{1}{N}\sum_{j=0}^{N-1}\alpha\left(1+\alpha^{-1}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi j}{N}+\varphi+\psi\right)}\right)\sum_{l\geqslant 0}\left(-\bar{\alpha}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi j}{N}+\varphi+\psi\right)}\right)^l\\
&= \alpha\frac{1}{N}\sum_{j=0}^{N-1}\left(1+\alpha^{-1}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi j}{N}+\varphi+\psi\right)}\right)\left(\sum_{l=0}^{N-2}(-\bar{\alpha})^l\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi jl}{N}+l(\varphi+\psi)\right)} + R_{N-2,j}(\alpha)\right)\\
&= \alpha\frac{1}{N}\sum_{l=0}^{N-2}\sum_{j=0}^{N-1}\left[(-\bar{\alpha})^l\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi jl}{N}+l(\varphi+\psi)\right)} + \frac{(-\bar{\alpha})^l}{\alpha}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi j(l+1)}{N}+(l+1)(\varphi+\psi)\right)}\right]\\
&\quad + \frac{\alpha}{N}\sum_{j=0}^{N-1}\left(1+\alpha^{-1}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi j}{N}+\varphi+\psi\right)}\right)R_{N-2,j}(\alpha)\\
&= \alpha\frac{1}{N}\sum_{l=0}^{N-2}\delta_{l,0}\sum_{j=0}^{N-1}(-\bar{\alpha})^l\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi jl}{N}+l(\varphi+\psi)\right)}\\
&\quad + \frac{\alpha}{N}\sum_{j=0}^{N-1}\left(1+\alpha^{-1}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi j}{N}+\varphi+\psi\right)}\right)R_{N-2,j}(\alpha)\\
&= \alpha + \frac{1}{N}\sum_{j=0}^{N-1}\left(\alpha+\mathrm{e}^{\mathrm{i}\left(\frac{2\pi j}{N}+\varphi+\psi\right)}\right)R_{N-2,j}(\alpha).
\end{aligned}
$$

To obtain the stated result we have to bound the second term in C^k-norm. We have

$$
R_{N-2,j}(\alpha) = \sum_{l\geqslant N-1}\left(-\bar{\alpha}\,\mathrm{e}^{\mathrm{i}\left(\frac{2\pi j}{N}+\varphi+\psi\right)}\right)^l = (-\bar{\alpha})^{N-1}\sum_{l\geqslant 0}(-\bar{\alpha})^l\left(\mathrm{e}^{\mathrm{i}\left(\frac{2\pi j}{N}+\varphi+\psi\right)}\right)^{l+N-1}
$$

and note that the sum still defines a function with radius of convergence one, so this function and its derivatives up to order k are uniformly bounded for $\alpha \in \mathbb{D}_r$ by some number $B_k > 0$, hence $\|R_{N-2,j}\|_{C^k} \leqslant kr^{N-1}B_k$. Since the function $F_j(\alpha) = \alpha + \mathrm{e}^{\mathrm{i}(\frac{2\pi j}{N}+\varphi+\psi)}$ multiplying $R_{N-2}(\alpha)$ has C^1-norm bounded by $2+r$ and all higher derivatives zero, we find

$$\begin{aligned}\left\| \frac{1}{N}\sum_{j=0}^{N-1} F_j \cdot R_{N-2,j} \right\|_{C^k} &\leqslant \max_{j\in[0,N-1]} \sum_{m=0}^{k} \left(\|F_j\|_{C^0} \|R_{N-2,j}\|_{C^k} + m\|F_j\|_{C^1} \|R_{N-2,j}\|_{C^{k-1}} \right) \\ &\leqslant \left(k + \frac{k(k+1)}{2} \right) \max_{j\in[0,N-1]} \|R_{N-2,j}\|_{C^k} \\ &\leqslant \frac{k^2(k+3)}{2}(2+r)r^{N-1}B_k.\end{aligned}$$

Thus, for fixed $r < 1$ and $k \geqslant 1$ and given $\varepsilon > 0$, we see that there exists an $N_0 > 0$ such that for all $N \geqslant N_0$

$$\|\alpha \mapsto z(\alpha, \psi, \theta) - \alpha\|_{C^k} < \frac{\varepsilon}{2} \quad \text{on } \mathbb{D}_r \times S^1 \times \Theta.$$

Next, since this function is C^∞ (and actually analytic w.r.t. $\alpha, \bar{\alpha}, \psi, \theta$) it follows that there exists an open neighborhood $U \supset \Theta$ such that

$$\|\alpha \mapsto z(\alpha, \psi, \theta) - \alpha\|_{C^k} < \varepsilon \quad \text{on } \mathbb{D}_r \times S^1 \times U,$$

which completes the proof. $\square$

5.4.3. Asynchronous equilibrium in the closed system. The Watanabe–Strogatz equations with functions f, g given by (24) depend on α, ψ (and θ as constant of motion) through the function $z(\alpha, \psi, \theta)$ in (19). This makes it difficult to find fixed points and analyze their stability. Our approach is to first make the assumption that $z = \alpha$, thereby 'closing the equation' to a simple form. Secondly, after we find a stable fixed point, we use lemma 5.6 to prove that it persists after reinserting the original function $z(\alpha, \psi, \theta)$.

Let us therefore start analyzing the equations (18) with f and g given by (24) but z replaced by α. In this case the system reduces to a skew-product flow; the equation for α does not depend on ψ anymore and becomes

$$\dot{\alpha} = -\frac{\sigma}{2}\left(\mathrm{e}^{-\mathrm{i}\delta} + \beta\,\mathrm{e}^{\mathrm{i}\delta}\right)\alpha^2 + \mathrm{i}(1-\beta)\alpha + \frac{\sigma}{2}\left(\beta|\alpha|^2\,\mathrm{e}^{-\mathrm{i}\delta} + \mathrm{e}^{\mathrm{i}\delta}\right). \tag{25}$$

with $0 < \delta < \pi/4$. The next proposition characterizes the asynchronous branch.

Theorem 5.7. *Let*

$$0 < \sigma < \sigma_{\mathrm{f}} := \frac{\beta - 1}{\sqrt{1 + 2\beta\,\cos(2\delta)}}. \tag{26}$$

Then equation (25) has an exponentially attracting fixed point α^{async} given by

$$|\alpha^{\mathrm{async}}| = \frac{\beta - 1 - \sqrt{(\beta-1)^2 - \sigma^2(1 + 2\beta\,\cos(2\delta))}}{\sigma(1 + 2\beta\,\cos(2\delta))} \quad \textit{and} \quad \arg(\alpha^{\mathrm{async}}) = -\frac{\pi}{2} + \delta. \tag{27}$$

Moreover, α^{async} is the unique branch of fixed points satisfying $\lim_{\sigma\to 0}\alpha^{\mathrm{async}} = 0$.

Proof. Representing equation (25) in polar coordinates, $\alpha = r\mathrm{e}^{\mathrm{i}\eta}$, we obtain

$$\dot{r} = \frac{\sigma}{2}(1 - r^2)\cos(\eta - \delta), \tag{28a}$$

$$\dot{\eta} = 1 - \beta - \sigma\beta r\sin(\eta + \delta) - \frac{\sigma}{2}\frac{1 + r^2}{r}\sin(\eta - \delta). \tag{28b}$$

Because of the product structure of equation (28a), we readily see that fixed points correspond to either $r = 1$ or $\cos(\eta - \delta) = 0$. Since we are looking for the branch α^{async} of fixed points that converge to zero as $\sigma \to 0$, we discard the solution $r = 1$. Hence we substitute $\cos(\eta - \delta) = 0$ and $\sin(\eta - \delta) = \pm 1$ in equation (28b) and obtain the condition

$$\pm(\beta - 1) + \sigma\beta r\cos(2\delta) + \frac{\sigma}{2}\frac{1 + r^2}{r} = 0.$$

Solving for r, we note that $\sin(\eta - \delta) = 1$ leads to negative solutions, so we find that $\eta = -\pi/2 + \delta$ and

$$r_{\pm} = \frac{(\beta - 1) \pm \sqrt{(\beta - 1)^2 - \sigma^2(1 + 2\beta\cos(2\delta))}}{\sigma(1 + 2\beta\cos(2\delta))}. \tag{29}$$

These fixed points exist for

$$\sigma < \sigma_{\mathrm{f}} = \frac{(\beta - 1)}{\sqrt{1 + 2\beta\cos(2\delta)}},$$

where the square root is well-defined since $\delta \in (0, \frac{\pi}{4})$. The branch r_- uniquely satisfies the condition that it converges to zero as $\sigma \to 0$.

Next we analyze its stability. Evaluating the Jacobian matrix of the system (28) at the fixed point we obtain

$$J\left(r_-, -\frac{\pi}{2} + \delta\right) = \begin{pmatrix} 0 & \frac{\sigma}{2}(1 - r_-^2) \\ a & -\beta\sigma r_- \sin(2\delta) \end{pmatrix} \quad \text{with} \quad = \frac{\sigma}{2}\left[1 + 2\beta\cos(2\delta) - \frac{1}{r_-^2}\right]. \tag{30}$$

Considering (29) and interpreting $(\beta - 1)$ and $\sigma\sqrt{1 + 2\beta\cos(2\delta)}$ as the sides of a triangle, we apply a triangle inequality estimate and conclude that

$$1 + 2\beta\cos(2\delta) < \frac{1}{r_-^2}.$$

Therefore, it follows that for $\delta \in (0, \frac{\pi}{4})$ and $\sigma > 0$ we have $a < 0$ and

$$\mathrm{tr}(J) < 0 \quad \text{and} \quad \det(J) > 0,$$

hence both eigenvalues of J have negative real part, which completes the proof. □

5.4.4. Below backward critical coupling. We are interested in the large β regime. In this case, $\sigma_{\mathrm{b}} \to 1$ as $\beta \to \infty$. The Jacobian eigenvalues at the asynchronous state α^{async} are complex and their imaginary part is proportional to β whereas their real parts converge to a constant depending on δ and σ. As a consequence, near the fixed point solutions spiral towards the fixed point with high frequency. We can also construct a Lyapunov function near α^{async} but its domain is rather small. This shows that the nonlinear picture for $|\alpha| \approx 0$ is slightly more intricate. To better handle the dynamics of α we average out the fast oscillatory behavior and analyze the averaged system.

Theorem 5.8. *Let z be the order parameter of the system (19) corresponding to initial conditions in an open neighborhood U of $\Theta \subset \mathbb{T}^N$. Given $\delta_0 \in (0, 1), 0 < \delta < \pi/4, 0 < \sigma < 1$ and $\varepsilon > 0$ small enough, there exist $N_0 \in \mathbb{N}$ and $\beta_0 > 0$ such that for all $N > N_0, \beta > \beta_0$, and $z(0) \in \mathbb{D}_{\delta_0}$ there is $T_0 > 0$ such that*

$$|z(t) - \alpha^{\text{async}}| \leqslant \varepsilon, \quad \textit{for all} \;\; t > T_0.$$

The proof of this theorem is presented in appendix A. Roughly speaking, it means that if we follow the synchronized manifold M_C decreasing slowly the intra-coupling strength once it looses stability the order parameter will drop and stay near zero. Thus, only the asynchronous branch is stable for $0 < \sigma < 1$ and large β.

5.4.5. Asynchronous invariant manifold. The final step is to construct the invariant manifold relative to the asynchronous dynamics. To this end we use a persistence argument over the equation of α and ψ.

Proposition 5.9. *Let σ strictly satisfy inequality (26). Given $\varepsilon > 0$ there exists N_0 such that for each $N \geqslant N_0$ there exists a NAIM with invariant boundary, M_{async}, such that the order parameter z on M_{async} satisfies*

$$|z - \alpha^{\text{async}}| \leqslant \varepsilon,$$

where α^{async} is given by (27).

Proof. To construct the asynchronous invariant manifold, we change to coordinates α, ψ using the Watanabe–Strogatz approach section 5.4.1 while viewing the θ_i's as fixed parameters. First consider the modified system (18) with α as argument of the functions f, g. Then the equation for α decouples and we find a stable fixed point $\alpha^{\text{async}} \in \mathbb{C}$, with $|\alpha^{\text{async}}| < 1$ and explicitly given by (27). Since the dynamics for ψ is neutral, it follows that $M_{\text{WS}} = \{\alpha = \alpha^{\text{async}},\ \psi \in S^1\} \subset \mathbb{D} \times S^1$ is an NAIM. Lemma 5.6 shows that given a $k \geqslant 1$ and a ball $B_r(\alpha^{\text{async}}) \subset \mathbb{D}$, there exists an N_0 and for all $N \geqslant N_0$ a neighborhood $U \subset \mathbb{T}^N$ of the uniformly distributed states Θ, such that the substitution of α by $z(\alpha, \psi, \theta)$ is a C^k-small perturbation of equation (18).

We choose $k = 1$ and by persistence of NAIMs, see [8, 10], the original system with the true dynamics for z also has an NAIM

$$\tilde{M}_{\text{WS}}(\theta) = \{\alpha = \alpha^{\text{async}} + g^{\text{async}}(\psi, \theta),\ \psi \in S^1\} \subset \mathbb{D} \times S^1 \tag{31}$$

C^1-close to M_{WS}. Thus, since U is precompact, we have uniformly for all $\theta \in U$ that $\|g^{\text{async}}\|_{C^1} \leqslant \varepsilon$ when N_0 is sufficiently large, and thus on $\tilde{M}_{\text{WS}}(\theta)$ also $|z - \alpha^{\text{async}}| \leqslant \varepsilon$ holds. Now we consider $\theta \in \mathbb{T}^N$ as dynamical variables again; that is, we lift the system according to (20) to coordinates (α, ψ, θ). Since the θ_i's have no dynamics, also

$$M_{\text{async}} = \bigcup_{\theta \in U} \tilde{M}_{\text{WS}}(\theta) \times \{\theta\} \subset \mathbb{D} \times S^1 \times \mathbb{T}^N \tag{32}$$

is an NAIM, but with boundary[7]. Since θ has no associated dynamics, the boundary is invariant. $\square$

6. Statement and proof of the main theorems

In this section, we state and prove the main result of this paper 6.2. First we must recapitulate the results we obtained in section 5.

Consider two star networks with a large number of leaves N. The first star we assume to be in a synchronous state (labeled $+$ and whose coordinates are labeled using $^+$) while the second network we assume to be in an asynchronous state (labeled $-$ and whose coordinates are labeled using $^-$). Both the synchronous and asynchronous states correspond to normally attracting invariant submanifolds of each star network, i.e., in terms of the objects we introduced so far:

- The $+$ star is in the synchronous invariant manifold given by $M_+ = M_C$ in ϕ_i^+, φ_0^+ coordinates with $i = 1, \ldots, N$, where M_C is defined in (9).
- The $-$ star is in the asynchronous invariant manifold given by $M_- = M_{\text{async}}$ in $\varphi_0^-, \alpha, \psi, \theta \in Q^- := S^1 \times \mathbb{D} \times S^1 \times \mathbb{T}^N$ coordinates, where M_{async} is defined in (32).

Note that the product system has $M = M_+ \times M_-$ as NAIM. Moreover, the order parameters $z_M^+ : \mathbb{T}^{N+1} \to \mathbb{C}$ and $z_M^- : Q^- \to \mathbb{C}$ are understood as the restriction on M and this is extended when we couple the two stars with intra-coupling ε guaranteeing the existence of chimera states as presented in theorem 6.1.

Theorem 6.1 (NAIM for weakly coupled stars). *Consider the coupled star network system (5). For any $h \in C^1$ and $\zeta > 0$, there exist $\varepsilon_0 > 0$, $\delta_0 \in (0, \pi/4)$, $N_0 > 0$, $\beta_0 > 0$ and an open set $\mathcal{I} \subset \mathbb{R}$ such that for any $N > N_0$, $\delta \in (0, \delta_0)$, $\beta > \beta_0$ and $0 < \varepsilon \leqslant \varepsilon_0$, $\sigma \in \mathcal{I}$ the system has a NAIM with boundary M_ε that is $\mathcal{O}(\varepsilon)$-close to $M = M_+ \times M_- \subset \mathbb{T}^{N+1} \times Q^-$ such that*

$$r^+_{M_\varepsilon}(\varphi_0^+, \phi_1^+, \ldots, \phi_N^+) > 1 - \zeta \quad \text{and} \quad r^-_{M_\varepsilon}(\alpha, \psi, \theta, \varphi_0^-) < \zeta.$$

The choice of coupling function h between each star affects the order of time these chimera states exist. So, we split the results in theorems 6.2 and 6.4.

Note that N_0 enters the proof through lemma 5.6, which was used to conclude that the asynchronous manifold M_- is an NAIM.

Theorem 6.2 (At least metastable chimera states).

(*a*) [*General coupling*] *If $h \in C^1$ in theorem 6.1 is arbitrary, there exists an open set $U \subset \mathbb{T}^{2(N+1)}$ of initial conditions $\theta \in U$ on M_ε such that the coupled dynamics stays in M_ε for a time of order $1/\varepsilon$, uniformly for $\theta \in U$. In other words, chimera states surely exist for a time of order $1/\varepsilon$.*

(*b*) [*Kuramoto–Sakaguchi coupling*] *In particular, if the coupling $h \in C^\infty$ is of the form*

$$h(\phi_1, \phi_2) = c_1 \sin(\phi_1 - \phi_2 + \delta).$$

Then there exists an open set $U \subset \mathbb{T}^{2(N+1)}$ of initial conditions $\theta \in U$ on M_ε such that the coupled dynamics stays in M_ε for a time of order $1/\varepsilon^2$, uniformly for $\theta \in U$.

[7] Here we mean the topological boundary, that is, $\partial M_{\text{async}} = \overline{M}_{\text{async}} \backslash M_{\text{async}}$.

If extra information on the coupling is given or the coupling structure is not fully symmetric, then the chimeras can be asymptotically stable. This motivates us to highlight and discuss how general is our results:

Remark 6.3. Let $A \in \mathbb{R}^{N+1\times N+1}$ be the adjacency matrix corresponding to the intercoupling between both stars. In a general flavor the networks dynamics we analyze is given as

$$\begin{aligned} \dot{\varphi}_0^+ &= \beta + \beta\frac{\sigma}{N}\sum_{j=1}^{N} \sin(\varphi_j^+ - \varphi_0^+ + \delta) + \varepsilon\sum_{j=0}^{N} A_{0j}\, h(\varphi_0^+ - \varphi_j^-), \\ \dot{\varphi}_i^+ &= 1 + \sigma \sin(\varphi_0^+ - \varphi_i^+ + \delta) + \varepsilon\sum_{j=0}^{N} A_{ij}\, h(\varphi_i^+ - \varphi_j^-), \quad i = 1, \ldots, N, \end{aligned} \tag{33}$$

and likewise for the '−' star. Both previous theorems, theorems 6.1 and 6.2-(a), are extended to this general coupling structure between stars without any extra condition, since the persistence argument goes through.

In the case of a small number of connections between both stars, we obtain

Theorem 6.4 (Stable chimera states). *Consider the coupled star network system* (33). *For any* $h \in C^1$ *and* $\zeta > 0$ *there exists* $\varepsilon_0 > 0$, $\delta_0 \in (0, \pi/4)$, $N_0 > 0$, $\beta_0 > 0$ *and an open set* $\mathcal{I} \subset \mathbb{R}$ *such that for any* $\beta > \beta_0$, $\delta \in (0, \delta_0)$, $N > N_0$ *and* $0 \leqslant k/N \ll 1$, *where only* k *of the leaves are coupled. That is,* $A_{ii} = 1$ *for* $i \in I$ *and* $A_{ij} = 0$ *otherwise, with* $|I| = k$. *Then for any* $0 < \varepsilon \leqslant \varepsilon_0$, $\sigma \in \mathcal{I}$ *and initial conditions on* M_ε *the system stays near a product of synchronous and asynchronous states for all time, even though it may leave* M_ε.

Note that here N_0 enters the proof in a similar way as in theorem 6.1, through the modified lemma 6.5. Essentially, we show that the coupling of nodes does not affect the constants of motion associated to the other nodes. And if the number k of coupled nodes is small, then a modified version of lemma 5.6 can still be used to conclude that M_- persists in the sense that its real order parameter r^- stays close to zero for all time.

6.1. Proof of theorem 6.1—NAIM for weakly coupled stars

Let $\mathcal{I} := (\sigma_\mathrm{b}, \sigma_\mathrm{f}) \subset \mathbb{R}$ with $\sigma_\mathrm{b}, \sigma_\mathrm{f}$ given by equation (8) and (26), respectively. It follows that $\mathcal{I}$ is open for $\beta > 1$. Fix $\sigma \in \mathcal{I}$ and δ_0 and β_0 such that hypothesis 5.3 is satisfied for all $\beta > \beta_0$ and $\delta \in (0, \delta_0)$. Thus, by proposition 5.4 we conclude that there exists an NAIM M_+.

Moreover, given $\zeta > 0$ sufficiently small, proposition 5.9 guarantees a sufficiently large N_0 such that there exists an NAIM M_- satisfying $|z_M^- - \alpha^{\mathrm{async}}| < \zeta/3$ for all $N > N_0$. Moreover, by theorem 5.7 for sufficiently large $\beta_0 > 1$ we obtain $|\alpha^{\mathrm{async}}| < \zeta/3$.

Consider the product manifold $M = M_+ \times M_-$ and fix an inter-coupling function $h \in C^1$. Since both M_+ and M_- are NAIMs that have exactly and approximately neutral dynamics, respectively, the product M is again an NAIM. When we couple the two stars we introduce a coupling of size ε between the two systems, with $\varepsilon \leqslant \varepsilon_0$ sufficiently small, then this perturbation of the product system will again have a NAIM M_ε close to M. This manifold can be described as

$$\begin{aligned} M_\varepsilon = \big\{ &\phi_i^+ = \phi^C + \tilde{g}_i^C(\varphi_0^\pm, \psi, \theta),\ \alpha = \alpha^{\mathrm{async}} \\ &+ (g^{\mathrm{async}} + \tilde{g}^{\mathrm{async}})(\varphi_0^\pm, \psi, \theta),\ \varphi_0^\pm, \psi \in S^1,\ \theta \in U \big\}, \end{aligned} \tag{34}$$

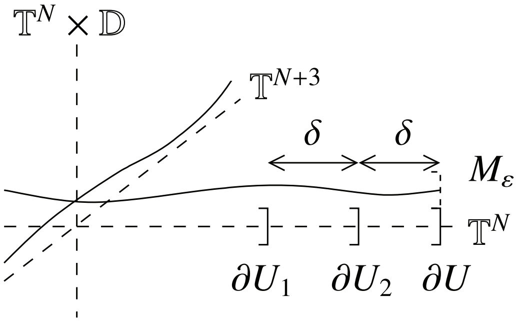

Figure 9. Flow along the invariant manifold M_ε.

with functions $\|\tilde{g}^C\|_{C^1}, \|\tilde{g}^{\text{async}}\|_{C^1} \in \mathcal{O}(\varepsilon)$ describing the perturbation of M_ε away from M. Note that since the combined system is invariant under a global phase shift, the functions $\tilde{g}^C, \tilde{g}^{\text{async}}$ only depend on $\varphi_0^\pm$ through their phase difference $\varphi_0^+ - \varphi_0^-$. Finally, since M_ε is an NAIM, it follows that M_ε has an invariantly foliated stable manifold $W^s(M_\varepsilon)$, see [10, theorem 4.1]. This implies that the orbits of points in a leaf $W^s(m)$ shadow and exponentially in time converge to the orbit of $m \in M_\varepsilon$.

Although M_ε has a boundary, both the persistence result and existence of the invariant stable foliation hold true on M with an arbitrarily small neighborhood of its boundary removed. That is, we modify the perturbed dynamics in a vertical neighborhood over ∂U, such that the modified coupled dynamics leaves a vertical section $S = \mathbb{T}^N \times \mathbb{D} \times \mathbb{T}^{N+3} \times \partial U$ over ∂M invariant. Hence persistence of M to M_ε is well-defined, where M_ε again has an invariant boundary with $\partial M_\varepsilon \subset S$, see figure 9. This follows from theorem 4.8 in [7, section 4.2]. Since M_ε is invariant, it again has an invariant stable foliation. As long as solutions of the coupled system on M_ε stay away from ∂M_ε, they satisfy the unmodified dynamics and the conclusions drawn above remain true.

Finally, we adjust ε_0 such that

$$\|\tilde{g}^C\|_{C^1} < \zeta/3 \quad \text{and} \quad \|\tilde{g}^{\text{async}}\|_{C^1} < \zeta/3$$

for all $0 < \varepsilon < \varepsilon_0$. Since the complex order parameter z^+ restricted to M_+ has absolute value 1 and $|z^+_{M_\varepsilon} - z^+_M| < \zeta$, we obtain that $r^+_{M_\varepsilon} > 1 - \zeta$. Likewise,

$$\begin{aligned} r^-_{M_\varepsilon} &= |z^-_{M_\varepsilon}| = |z^-_{M_\varepsilon} - z^-_M + z^-_M + \alpha^{\text{async}} - \alpha^{\text{async}}| \\ &\leqslant |z^-_{M_\varepsilon} - z^-_M| + |z^-_M - \alpha^{\text{async}}| + |\alpha^{\text{async}}| \\ &< \zeta. \end{aligned} \tag{35}$$

This proves theorem 6.1.

6.2. *Proof of theorem 6.2 (a)—general coupling*

For point (a) of theorem 6.2, we have to study how the coupling may perturb the trivial dynamics $\dot{\theta} = 0$ in the asynchronous system, and drive the system across the boundary of U, where normal hyperbolicity may be lost. In view of (34), we can express the invariant manifold M_ε for the coupled system as a graph, where the coordinates ϕ_i^+ and α depend on the other coordinates $\varphi_0^+, \varphi_0^-, \psi, \theta \in \mathbb{T}^{N+3} \times U$, see figure 9. Now the boundary of M_ε is contained in the vertical section $S = \mathbb{T}^N \times \mathbb{D} \times \mathbb{T}^{N+3} \times \partial U$ over $\mathbb{T}^{N+3} \times \partial U$. Solution curves that leave M_ε do so by θ leaving U through its boundary ∂U.

In the uncoupled setup there was no dynamics along $\theta \in U \subset \mathbb{T}^N$, that is, $\dot{\theta} = 0$, thus, with coupling we have $\dot{\theta} \in \mathcal{O}(\varepsilon)$. We choose open neighborhoods U_1, U_2 such that

$$\Theta \subset U_1 \subset U_2 \subset U \subset \mathbb{T}^N \quad \text{with } B(U_1, \delta) \subset U_2 \text{ and } B(U_2, \delta) \subset U$$

for some fixed $\delta > 0$, see also figure 9. We may also assume that the modifications we made for theorem 6.1 are contained outside U_2. Solution curves with initial conditions in M_ε and $\theta \in U_1$ will take $\mathcal{O}\left(\frac{\delta}{\varepsilon}\right)$ time to cross the gap $U_2 \backslash U_1$. As the vector field is unmodified in this part of phase space, these conclusions are true for the unmodified system. This proves point (a).

6.3. *Proof of theorem 6.2 (b)—Kuramoto–Sakaguchi coupling*

We continue to prove point (b) of theorem 6.2. Consider equation (5) in the ϕ coordinates and assume that the coupling is sinusoidal, that is, $h = c_1 \sin + c_2 \cos$. Thus the equations for ϕ^- become

$$\dot{\phi}_i^- = (1-\beta) - \sigma \sin(\phi_i^- - \delta) - \beta \frac{\sigma}{N} \sum_{j=1}^{N} \sin(\phi_j^- + \delta) + \varepsilon\, h(\phi_i^+ - \phi_i^- + \varphi_0^+ - \varphi_0^-). \quad (36)$$

Since we know already that the Watanabe–Strogatz approach can be applied to the first terms comprising the uncoupled system, we focus on the coupling term $h(\phi_i^+ - \phi_i^- + \varphi_0^+ - \varphi_0^-)$. We denote the associated vector field by $X \in \mathfrak{X}(\mathbb{T}^N)$, but note that X also depends on $\phi^+, \varphi_0^\pm$ as external variables. We can write this as

$$h(\phi_i^+ - \phi_i^- + \varphi_0^+ - \varphi_0^-) = h_1(-\phi_i^-) h_2(\phi_i^+ + \varphi_0^+ - \varphi_0^-),$$

where h_1, h_2 again denote sinusoidal functions. We cannot apply the Watanabe–Strogatz lift (15) to X, since $h_2(\phi_i^+ + \varphi_0^+ - \varphi_0^-)$ is not identical for all i. However, when we restrict to the invariant manifold M_ε, then $\phi_i^+ = \phi^C + \tilde{g}_i^C(\varphi_0^+ - \varphi_0^-, \psi, \theta)$ with $\tilde{g}_i^C \in \mathcal{O}(\varepsilon)$. Hence we find

$$h(\phi_i^+ - \phi_i^- + \varphi_0^+ - \varphi_0^-) = h_1(-\phi_i^-) h_2(\phi^C + \varphi_0^+ - \varphi_0^-) + \mathcal{O}(\varepsilon).$$

Now we can lift the vector field X' associated to $h_1(\phi_i^-) h_2(\phi^C + \varphi_0^+ - \varphi_0^-)$ using the Watanabe–Strogatz lift (15). This implies that $\hat{X}'$ leaves θ constant, and since the remaining part of the coupling vector field is of order ε^2, it follows that $\dot{\theta} \in \mathcal{O}(\varepsilon^2)$. This proves claim 2 of theorem 6.2.

6.4. *Proof of theorem 6.4—Stable chimera states*

Finally, we prove theorem 6.4. We now assume that the coupling matrices are of the form $A_{ii} = 1$ when $i \in I \subset \mathbb{N}$ and zero otherwise, where $|I| = k \ll N$. The coupling function $h \in C^1$ is arbitrary; since h is defined on a compact set, $\|h\|_{C^1}$ is bounded.

When only a small fraction of the leaves of the two stars are coupled, we can improve the previous argument to obtain that the chimera state is fully stable for all time. To this end, we apply the two different lifts of the vector field for the asynchronous star. We apply the Watanabe–Strogatz lift to the uncoupled vector field and the lift (23) to the coupling terms.

For $\varepsilon = 0$, the dynamics of the asynchronous star are given in Watanabe–Strogatz lifted coordinates by (18) and (19) and $\dot{\theta} = 0$. The extra coupling term is given in ϕ coordinates by

$$\dot{\phi}_i^\pm = \varepsilon\, h(\phi_i^\mp - \phi_i^\pm \pm \Gamma) \quad \text{for } i \in I, \quad (37)$$

where $\Gamma = \varphi_0^+ - \varphi_0^-$. Let us denote by $\varepsilon Y \in \mathfrak{X}(\mathbb{T}^N)$ the vector field describing the evolution of the ϕ_i^- variables according to (37). We use (23) to find the lift $\hat{Y} \in \mathfrak{X}(\mathbb{D} \times S^1 \times \mathbb{T}^N)$ of Y that gives the dynamics in Watanabe–Strogatz coordinates. We calculate $G'_{\alpha,\psi}(\theta) = \frac{\partial \phi}{\partial \theta}(\alpha, \psi, \theta)$ with $\phi = \pi(\alpha, \psi, \theta)$. From $\mathrm{e}^{\mathrm{i}\phi} = G_{\alpha,\psi}(w)$ with $w = \mathrm{e}^{\mathrm{i}\theta}$ we obtain

$$\begin{aligned}
\mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi}\frac{\partial \phi}{\partial \theta} &= \frac{\partial}{\partial w}\left[\frac{\alpha + \mathrm{e}^{\mathrm{i}\psi} w}{1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi} w}\right]\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta} \\
\iff \quad \frac{\alpha + \mathrm{e}^{\mathrm{i}\psi} w}{1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi} w}\frac{\partial \phi}{\partial \theta} &= \left[\frac{\mathrm{e}^{\mathrm{i}\psi}}{1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi} w} - \frac{\alpha + \mathrm{e}^{\mathrm{i}\psi} w}{(1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi} w)^2}\bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi}\right] w \\
\iff \quad (\alpha + \mathrm{e}^{\mathrm{i}\psi} w)\frac{\partial \phi}{\partial \theta} &= \frac{(1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi} w)\,\mathrm{e}^{\mathrm{i}\psi} w - \alpha\bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi} w - \bar{\alpha}\,\mathrm{e}^{2\mathrm{i}\psi} w^2}{1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi} w} \\
\iff \quad \frac{\partial \phi}{\partial \theta}(\alpha, \psi, \theta) &= \frac{(1 - \alpha\bar{\alpha})\,\mathrm{e}^{\mathrm{i}(\psi+\theta)}}{(1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}(\psi+\theta)})(\alpha + \mathrm{e}^{\mathrm{i}(\psi+\theta)})} = \frac{1 - |\alpha|^2}{|\alpha + \mathrm{e}^{\mathrm{i}(\psi+\theta)}|^2}.
\end{aligned}$$

Thus we find that $\hat{Y}$ is given by

$$\begin{aligned}
\dot{\theta}_i &= \varepsilon\left(\frac{\partial \phi}{\partial \theta}\right)^{-1} \cdot h\left(\phi_i^- - \phi_i^+ - \Gamma\right) \\
&= \varepsilon\,\frac{|\alpha + \mathrm{e}^{\mathrm{i}(\psi+\theta_i)}|^2}{1 - |\alpha|^2}\,h\left(\pi_i(\alpha, \psi, \theta_i) + \varphi_0^- - \varphi_i^+\right) \qquad \text{for } i \in I,
\end{aligned} \tag{38}$$

$\dot{\theta}_i = 0$ for $i \notin I$, $\dot{\alpha} = 0$ and $\dot{\psi} = 0$. We use the fact that the θ_i with $i \notin I$ are still conserved. With modified version of lemma 5.6 we show that if $k \ll N$, then the result of the lemma still holds true with the values of θ_i with $i \in I$ arbitrary, hence there exists an invariant manifold with invariant boundary for the asynchronous star network.

For simplicity, let us assume by relabelling that $I = \{N - k + 1, \ldots, N\}$. Define the set

$$\bar{\Theta} := \left\{(\vartheta, \xi) \in \mathbb{T}^{N-k} \times \mathbb{T}^k | \exists \theta \in \Theta : \forall 1 \leqslant i \leqslant N - k : \vartheta_i = \theta_i\right\} \subset \mathbb{T}^N. \tag{39}$$

Note that $\bar{\Theta}$ is a simple extension of Θ with the first $N - k$ entries still uniformly distributed according to definition 5.5 and the last k arbitrary. The coupled dynamics leaves $\bar{\Theta}$ invariant since $\dot{\vartheta} = 0$, still. We now have the following modified version of lemma 5.6; here we fix a C^1-norm, but this is all we needed anyways.

Lemma 6.5. *Fix $0 < r < 1$ and consider the disc $\mathbb{D}_r = \{\alpha \in \mathbb{C} | |\alpha| < r\}$. For each $\varepsilon > 0$ there exists for all N sufficiently large and $k \ll N$ an open neighborhood $U = U' \times \mathbb{T}^k$ of $\bar{\Theta} \subset \mathbb{T}^N$ such that the map $\mathbb{D}_r \to \mathbb{C},\ \alpha \mapsto z(\alpha, \psi, \theta) - \alpha$, with $z(\alpha, \psi, \theta)$ given by (19), is smaller than ε in C^k-norm, uniformly for all $(\psi, \theta) \in S^1 \times U$.*

Proof. The proof is a straightforward extension of the proof of lemma 5.6. Given $\theta' = (\vartheta, \xi) \in \bar{\Theta}$, let $\theta \in \Theta$ as in (39). For $z(\alpha, \psi, \theta')$ we can estimate the difference with $z(\alpha, \psi, \theta)$ as

$$z(\alpha, \psi, \theta') - z(\alpha, \psi, \theta) = \frac{1}{N}\sum_{j=N-k+1}^{N}\left(\frac{\alpha + \mathrm{e}^{\mathrm{i}\psi}\,\mathrm{e}^{\mathrm{i}\xi_j}}{1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi}\,\mathrm{e}^{\mathrm{i}\xi_j}} - \frac{\alpha + \mathrm{e}^{\mathrm{i}\psi}\,\mathrm{e}^{\mathrm{i}\theta_j}}{1 + \bar{\alpha}\,\mathrm{e}^{\mathrm{i}\psi}\,\mathrm{e}^{\mathrm{i}\theta_j}}\right).$$

Since $|\alpha| < r$ is bounded away from 1, all terms are uniformly bounded in C^1-norm, say by B. Hence $\|z(\alpha,\psi,\theta') - z(\alpha,\psi,\theta)\|_{C^1} \leqslant \frac{k}{N}B$, and thus for $k \ll N$ this yields a sufficiently small contribution to the overall C^1-norm of $\alpha \mapsto z(\alpha,\psi,\theta') - \alpha$ relative to the estimate for $z(\alpha,\psi,\theta)$ in the proof of lemma 5.6. □

To prove theorem 6.4, we now use lemma 6.5 instead of lemma 5.6. Note that $U = U' \times \mathbb{T}^k$ is invariant under the coupled dynamics since its boundary $\partial U = \partial U' \times \mathbb{T}^k$ is invariant. Thus, we find that also the persistent manifold M_ε has an invariant boundary.

6.5. Coupling through hubs

Connecting two stars using a general coupling mechanism invalidates the use of the Watanabe–Strogatz approach and the description of the dynamics through the action of the Möbius group. An important exception to this rule is when only the hubs of the stars are coupled, since, as we will show, we can still apply the WS dimensional reduction method, even if now we have that the phase difference of the hubs acts as an *external field* to the equations describing the order parameters z.

Consider two stars with their hubs coupled sinusoidally, with strength ε,

$$\begin{aligned}
\dot{\varphi}_k^{\pm} &= 1 + \sigma\,\sin(\varphi_0^{\pm} - \varphi_k^{\pm} + \delta),\\
\dot{\varphi}_0^{\pm} &= \beta + \frac{\sigma\beta}{N}\sum_{j=1}^{N}\,\sin(\varphi_j^{\pm} - \varphi_0^{\pm} + \delta) + \varepsilon\,\sin(\varphi_0^{\mp} - \varphi_0^{\pm} + \delta),
\end{aligned} \tag{40}$$

Again we change to coordinates $\phi_k^{\pm} = \varphi_k^{\pm} - \varphi_0^{\pm}$ that measure the relative phases of the leaves with respect to the hubs. Secondly, we introduce $\Gamma = \varphi_0^{+} - \varphi_0^{-}$, the difference between the hub phases. With these new variables, the equations become

$$\dot{\phi}_k^{\pm} = 1 - \beta - \sigma\,\sin(\phi_k^{\pm} - \delta) - \frac{\beta\sigma}{N}\sum_{j=1}^{N}\,\sin(\phi_j^{\pm} + \delta) + \varepsilon\,\sin(\mp\Gamma + \delta),$$

$$\dot{\Gamma} = \frac{\sigma\beta}{N}\sum_{j=1}^{N}\left(\sin(\phi_k^{+} + \delta) - \sin(\phi_k^{-} + \delta)\right) - 2\varepsilon\,\cos(\delta)\,\sin(\Gamma).$$

For each star the equations of motion are in the appropriated form and we can apply the WS approach, where the functions $f^{\pm}$ and $g^{\pm}$ are now given by

$$\begin{aligned}
f^{\pm} &= \frac{-\sigma\,\mathrm{e}^{-\mathrm{i}\delta}}{2i},\\
g^{\pm} &= 1 - \beta - \beta\sigma\left(\frac{z^{\pm}\,\mathrm{e}^{\mathrm{i}\delta} - \overline{z}^{\pm}\,\mathrm{e}^{-\mathrm{i}\delta}}{2i}\right) + \varepsilon\,\sin(\mp\Gamma + \delta).
\end{aligned}$$

The difference here is that the functions $g^{\pm}$ now depend on the phase difference Γ of the hubs, that, in its turn, depends on the relative phases of the leaves of both populations. Nevertheless, the dependence is the same for all leaves and therefore follows the WS approach. The equations for the WS variables $\alpha^{\pm}$ are thus given by

$$\dot{\alpha}^{\pm} = -\frac{\sigma}{2}(\mathrm{e}^{-\mathrm{i}\delta} + \beta\,\mathrm{e}^{\mathrm{i}\delta})(\alpha^{\pm})^2 + i(1-\beta)\left(1 + \varepsilon\frac{\sin(\mp\Gamma + \delta)}{1-\beta}\right)\alpha^{\pm} + \frac{\sigma}{2}\beta\,\mathrm{e}^{-\mathrm{i}\delta}|\alpha^{\pm}|^2 + \frac{\sigma}{2}\,\mathrm{e}^{\mathrm{i}\delta},$$

where we used that $z^{\pm} = \alpha^{\pm}$. The difference now is that where before would appear only the natural frequency of the leaves $\omega = 1$, now we have a term $1 + \varepsilon \frac{\sin(\mp\Gamma+\delta)}{1-\beta}$ that comes from the coupling. The equation for Γ can be written as

$$\dot{\Gamma} = \sigma\beta\, \Im((z^{+} - z^{-})\mathrm{e}^{-\mathrm{i}\delta}) - 2\varepsilon\, \cos(\delta)\, \sin(\Gamma)$$

We can view the effect of the coupling as inducing oscillations of amplitude $\varepsilon/(1-\beta)$ to the natural frequency ω and therefore we expect the persistence of the fixed points.

7. Open problems and conclusions

We presented chimera states born from the coexistence of synchronous and asynchronous dynamics in the mean-field of a single star graph. We showed that the chimera states in coupled star graphs correspond to the existence of an invariant manifold with boundary. We associated their breaking with dynamics running into the boundary of the invariant manifold and provided lower bounds for their survival. Predicting whether such chimeras break to a complete synchronous dynamics of the network or to asynchronous remains an open problem.

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Appendix A. Proof of discontinuous transition theorem 5.8

The averaging principle will play role in our analysis. Let $D \subset \mathbb{R}^n$ be an open set. $F_1 \in C^1(D, S^1)$, Ω and F_2 are Lipschitz and all three functions are 2π-periodic in η. Moreover, Ω is bounded away from zero, i.e.,

Hypothesis A.1. Suppose

$$0 < m_1 \leqslant \inf_{(x,\eta)\in D\times S^1} |\Omega(x,\eta)| \leqslant \sup_{(x,\eta)\in D\times S^1} |\Omega(x,\eta)| \leqslant m_2 < \infty$$

where m_1 and m_2 are ε-independent constants.

Consider the *original* system given by

$$\begin{aligned} \dot{x} &= \varepsilon F_1(x,\eta), \\ \dot{\eta} &= \Omega(x,\eta) + \varepsilon\, F_2(x,\eta), \quad (x(0),\eta(0)) = (x_0,\eta_0),\ x \in D,\ \eta \in S^1. \end{aligned} \tag{A.1}$$

and the *averaged* system given by

$$\begin{aligned} \dot{y} &= \varepsilon \bar{F}(y), \\ \dot{\chi} &= \Omega(y,\chi), \quad (y(0),\chi(0)) = (x_0,\chi_0) \in D \times S^1, \end{aligned} \tag{A.2}$$

where

$$\bar{F}(y) = \frac{\frac{1}{2\pi}\int_0^{2\pi} \frac{F_1(y,\vartheta)}{\Omega(y,\vartheta)}\mathrm{d}\vartheta}{\frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\vartheta}{\Omega(y,\vartheta)}}$$

is the first order averaged vector field. If the averaged system has an asymptotically stable fixed point, solutions of the original system are approximated by averaged ones. That is, we have the following adapted version of the theorem, see [6, theorem 7.2] or [22, lemma 7.10.1–2, theorem 7.10.4].

Theorem A.2 (Averaging with attraction). *Let $D \subset \mathbb{R}^n$ be an open set and consider the equation (A.1). Suppose that the averaged system equation (A.2) has an asymptotically stable fixed point $y = 0, \bar{F} \in C^1(D)$ and has a domain of attraction $D_0 \subset D$. For any compact $K \subset D_0$ there exists a ε_0 and $c > 0$ such that for all $x_0 \in K$ and for each $\varepsilon < \varepsilon_0$*

$$\|x(t) - y(t)\| \leqslant c\varepsilon, \quad 0 \leqslant t < \infty.$$

Now, we present the proof of theorem 5.8. We will obtain this result as a consequence of three lemmas. To this end, we first need to establish some preliminaries.

The planar system (28) in polar coordinates can be rewritten in the form

$$\dot{r} = F_1(r,\eta) \quad \text{and} \quad \dot{\eta} = \beta\ \Omega(r,\eta) + F_2(r,\eta)$$

where

$$F_1(r,\eta) := \frac{\sigma}{2}(1 - r^2)\ \cos(\eta - \delta), \tag{A.3a}$$

$$\Omega(r,\eta) := -1 - \sigma r\ \sin(\eta + \delta), \tag{A.3b}$$

$$F_2(r,\eta) := 1 - \frac{\sigma}{2}\frac{1 + r^2}{r}\ \sin(\eta - \delta). \tag{A.3c}$$

We can parameterize time using the parameter β and introduce an averaged system on the new time $\tau := \beta t$. First, we need some auxiliary results.

Lemma A.3. *Fix $\delta_0 \in (0, 1)$ and consider $R : [\delta_0 - 1, 1 - \delta_0] \to \mathbb{R}$ as*

$$R(x) = \frac{P(x)}{Q(x)},$$

where

$$P(x) = -\frac{1}{\pi}\int_0^{2\pi} \frac{\sin\vartheta}{1 + x\ \sin\vartheta}\mathrm{d}\vartheta \quad \textit{and} \quad Q(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\vartheta}{1 + x\ \sin\vartheta}.$$

Then R is a smooth function satisfying

$$R(0) = 0, R(x) > 0 \quad \textit{for}\ x \in (0, 1 - \delta_0], \quad \textit{and} \quad \frac{\mathrm{d}R(0)}{\mathrm{d}x} = 1.$$

Proof. First we calculate $P(x)$. We Taylor expand the function $1/(1 + x \sin\vartheta)$ to obtain

$$P(x) = -\frac{1}{\pi}\int_0^{2\pi} \sin\vartheta \sum_{n=0}^{\infty}(-1)^n(x\ \sin\vartheta)^n\mathrm{d}\vartheta$$

notice that $\int_0^{2\pi} \sin^{2k+1}\vartheta \mathrm{d}\vartheta = 0$ as $\sin^{2k+1}\vartheta$ is an odd function in $[0, 2\pi]$ and thus

$$-\frac{1}{\pi}\int_0^{2\pi} \frac{\sin(\vartheta)}{1 + x\,\sin(\vartheta)} \mathrm{d}\vartheta = \sum_{k=1}^{\infty} a_{2k} x^{2k-1}$$

where

$$a_{2k} = \frac{1}{\pi}\int_0^{2\pi} \sin^{2k}\vartheta \mathrm{d}\vartheta > 0, \quad k \geqslant 1$$

so P is a monotone function. Moreover,

$$a_2 = 1, \qquad a_4 = \frac{3}{4}, \quad \text{and} \quad a_6 = \frac{5}{8}.$$

Repeating the same procedure

$$Q(x) = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty}(-1)^n(x\,\sin\vartheta)^n \mathrm{d}\vartheta = 1 + \sum_{k=1}^{\infty}\frac{a_{2k}}{2}x^{2k} = 1 + \frac{x}{2}P(x).$$

Therefore, $R(x)$ is positive, and the claim follows. □

This lemma will play a role in the averaging approximation for the dynamics of α.

Lemma A.4. *Fix $0 < \delta_0 < 1$ and consider $\mathbb{D}_{\delta_0} := \{z \in \mathbb{C} : |z| < 1 - \delta_0\}$. For any $0 < \sigma < 1$, $0 < \delta < \pi/4$, $\alpha_0 \in \mathbb{D}_{\delta_0}$ there exist constants $c > 0$, and $\beta_0 > 0$ such that for each $\beta > \beta_0$ the solutions $\alpha(t)$ of equation (A.3) in $\tau := \beta t$ with initial condition $\alpha_0 \in \mathbb{D}_{\delta_0}$ satisfy*

$$\|r(\tau) - \rho(\tau)\| < \frac{c}{\beta} \quad \text{for } \tau \geqslant 0, \tag{A.4}$$

where $\rho(\tau)$ is the trajectory of the averaged system

$$\rho' = \frac{1}{\beta}\bar{F}(\rho) \tag{A.5}$$

with initial condition $\rho(\eta_0) = r_0$, where $\bar{F} : [\delta_0 - 1, 1 - \delta_0] \to \mathbb{R}$ is smooth and satisfying

$$\bar{F}(0) = 0, \bar{F}(\rho) < 0 \quad \textit{for } \rho \in (0, 1 - \delta_0] \quad \textit{and} \quad \frac{\mathrm{d}\bar{F}(0)}{\mathrm{d}\rho} = -\frac{1}{4}\sigma\,\sin 2\delta.$$

Proof. Performing a change of time scale $t \mapsto \tau = \beta t$ in equation (A.3), we obtain:

$$r' = \varepsilon\, F_1(r, \eta) \quad \text{and} \quad \eta' = -1 - \sigma r\,\sin(\eta + \delta) + \varepsilon F_2(r, \eta), \tag{A.6}$$

where $'$ denotes differentiation with respect to τ and $\varepsilon = 1/\beta$. The transformation induces a slow–fast structure where the small parameter is ε. Notice that $r = 1$ is a fixed point of r and thus solutions starting with $r < 1$ cannot have $r > 1$ and neither have $r = 1$ in finite time. Thus, η' is bounded away from zero, hypothesis A.1, and we can use η as our new time. To apply the averaging principle, we define the averaged vector field as

$$\bar{F}(r) := \frac{\frac{1}{2\pi}\int_0^{2\pi} H(r, \vartheta, 0)\mathrm{d}\vartheta}{\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\vartheta}{-1-\sigma r\,\sin(\vartheta+\delta)}}, \tag{A.7}$$

where

$$H(r,\eta,\varepsilon) = \frac{\frac{\sigma}{2}(1-r^2)\cos(\eta-\delta)}{-[1+\sigma r\sin(\eta+\delta)] + \varepsilon\left(1-\frac{\sigma}{2}\frac{1+r^2}{r}\sin(\eta-\delta)\right)}. \tag{A.8}$$

The denominator of equation (A.7) resembles the $Q(x)$ function of lemma A.3. So, replacing equation (A.8) into (A.7), we calculate the numerator:

$$\int_0^{2\pi} \frac{\cos(\eta-\delta)}{1+\sigma r\sin(\eta+\delta)}\,\mathrm{d}\eta = \cos(2\delta)\int_\delta^{2\pi+\delta} \frac{\cos(\vartheta)}{1+\sigma r\sin(\vartheta)}\mathrm{d}\vartheta$$
$$+\sin(2\delta)\int_\delta^{2\pi+\delta} \frac{\sin(\vartheta)}{1+\sigma r\sin(\vartheta)}\mathrm{d}\vartheta,$$

where we used the change of variables $\vartheta = \eta+\delta$ and the trigonometric relation for the cosine. Since both integrals are along a full cycle of ϑ, the first integral is zero. Then

$$-\frac{1}{\pi}\int_0^{2\pi} \frac{\cos(\eta-\delta)}{1+\sigma r\sin(\eta+\delta)}\mathrm{d}\eta = \sin(2\delta)P(\sigma r).$$

From this observation we obtain the vector field for ρ. Thus we obtain

$$\bar{F}(\rho) = -\frac{\sin 2\delta}{4}\sigma(1-\rho^2)R(\sigma\rho).$$

From lemma A.3 it follows that since $R(\sigma\rho) > 0$ for $\rho > 0$ we obtain $\bar{F}(\rho) < 0$.

To conclude the stability, notice that since $\bar{F} < 0$ for $\rho < 1$ we have that solutions converge to the origin. Solutions will enter a ball of radius δ_1 sufficiently small. We show that this convergence is exponential as $\frac{\mathrm{d}F^1(0)}{\mathrm{d}\rho} < 0$. The result follows by the principle of linearized stability: given $\delta_1 > 0$ small enough, all solutions with $\rho < \delta_1$ converge to the origin exponentially fast. Next applying averaging theorem A.2 our claim follows. $\square$

Lemma A.5. *Fix* $0 < \delta_0 < 1$. *Then, for any* $0 < \sigma < 1, 0 < \delta < \pi/4, \alpha_0 \in \mathbb{D}_{\delta_0}$, *there exists* $\beta_0 > 0$ *such that for each* $\beta > \beta_0$ *the solutions* $\alpha(t)$ *of equation* (A.3) *with initial condition* $\alpha_0 \in \mathbb{D}_{\delta_0}$ *converge to* α^{async}.

Proof. Given $\varepsilon_0 > 0$ small enough, lemma A.4 shows that there exist constant c and β_0 such that for every $\beta > \beta_0$ we have $\|\alpha(\tau)\| = \|r(\tau)\| \leqslant \varepsilon_0$, since 0 is a hyperbolic fixed point for ρ. Equation (A.3) has a stable fixed point α^{async} satisfying $|\alpha^{\text{async}}| \to 0$ as $\beta \to \infty$ as can be observed in equation (29). Moreover, when $\beta \to \infty$ the real part of the Jacobian eigenvalues of α^{async} tends to a constant independent of β. Thus, by the principle of linearized stability there is $\delta_I > 0$ independent of β such that any initial condition in the open ball $B(\alpha^{\text{async}}, \delta_I)$ converges to α^{async}. Now we choose $\varepsilon_0 < \delta_I$. This is guaranteed because δ_I is independent of β. Thus, when β is large enough, solutions α starting in $\mathbb{D}_{\delta_0}$ enter $B(\alpha^{\text{async}}, \delta_I)$ and thus must converge to α^{async}. $\square$

Proof of Theorem 5.8 By lemma 5.6 there is N_0 such that for all $N > N_0$

$$|z(\tau) - \alpha(\tau)| \leqslant \varepsilon/2.$$

Next, by the averaging principle there exists c and β_0 such that for all $\beta > \beta_0$ we find constants $M(\beta)$ and $\mu(\beta)$ such that

$$\begin{aligned}|\alpha(\tau) - \alpha^{\text{async}}| &\leqslant |r(\tau) - \rho(\tau)| + 2|\rho(\tau)| + |\rho(\tau) - r_-|\\ &\leqslant \frac{c}{\beta} + 2M\,|\rho_0|\,\mathrm{e}^{-\mu\tau} + M\,|\rho_0 - r_-|\,\mathrm{e}^{-\mu\tau}.\end{aligned}$$

Consequently, after a fixed finite time T_0 we can make it below $\varepsilon/2$. Applying the triangle inequality for $t > T_0$ we obtain

$$|z(t) - \alpha^{\text{async}}| \leqslant |z(t) - \alpha(t)| + |\alpha(t) - \alpha^{\text{async}}|$$

and the claim follows. □

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