Bifurcation Theory
Warming-up

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Prove the following results

Theorem 1 *Let* $(\mathbb{M}, \mathrm{d})$ *be a complete metric space and let* $T : \mathbb{M} \to \mathbb{M}$ *be a contraction mapping with contraction constant* k*. Then* T *has a unique fixed point* $x \in \mathbb{M}$*. Furthermore, if* $y \in \mathbb{M}$ *is arbitrarily chosen, then the iterates* $\{x_n\}_{n=0}^{\infty}$*, given by*

$$x_0 = y \quad (1)$$

$$x_n = T(x_{n-1}),\ n \geq 1, \quad (2)$$

have the property that $\lim_{n\to\infty} x_n = x$*.*

Theorem 2 *Let* $(\mathbb{M}, \mathrm{d})$ *be a complete metric space and let*

$$B = \{x \in \mathbb{M} : \mathrm{d}(z, x) < \epsilon\},$$

where $z \in \mathbb{M}$ *and* $\epsilon > 0$ *is a positive number and let* $T : B \to \mathbb{M}$ *be a mapping such that*

$$\mathrm{d}(T(y), T(x)) \leq k\mathrm{d}(x, y),\ \forall\ x, y \in B,$$

with contraction constant $k < 1$*. Furthermore assume that*

$$\mathrm{d}(z, T(z)) < \epsilon(1 - k).$$

Then T *has a unique fixed point* $x \in B$*.*

Theorem 3 *Let* $(\mathbb{M}, \mathrm{d})$ *be a metric space and let* $T : \mathbb{M} \to \mathbb{M}$ *be a mapping such that*

$$\mathrm{d}(T(x), T(y)) < \mathrm{d}(x, y), \quad \forall x, y \in \mathbb{M},\ x \neq y.$$

Furthermore assume that there exists $z \in \mathbb{M}$ *such that the iterates* $\{x_n\}_{n=0}^{\infty}$*, given by*

$$x_0 = z \tag{3}$$

$$x_n = T(x_{n-1}),\ n \geq 1, \tag{4}$$

have the property that there exists a subsequence $\{x_{n_j}\}_{j=0}^{\infty}$ *of* $\{x_n\}_{n=0}^{\infty}$*, with*

$$\lim_{j\to\infty} x_{n_j} = y \in \mathbb{M}.$$

Then y *is a fixed point of* T *and this fixed point is unique.*

Theorem 4 *Let* $(\mathbb{M}, \mathrm{d})$ *be a metric space and let* $T : \mathbb{M} \to \mathbb{M}$ *be a mapping such that*

$$\mathrm{d}(T(x), T(y)) < \mathrm{d}(x, y), \quad \forall x, y \in \mathbb{M},\ x \neq y.$$

Further assume that

$$T : \mathbb{M} \to K,$$

where K *is a compact subset of* $\mathbb{M}$*. Then* T *has a unique fixed point in* $\mathbb{M}$*.*

Theorem 5 *Let* $(\mathbb{M}, \mathrm{d})$ *be a complete metric space and let* $T : \mathbb{M} \to \mathbb{M}$ *be a mapping such that*

$$\mathrm{d}(T^m(x), T^m(y)) \leq k\mathrm{d}(x, y), \quad \forall x, y \in \mathbb{M},$$

for some $m \geq 1$*, where* $0 \leq k < 1$ *is a constant. Then* T *has a unique fixed point in* $\mathbb{M}$*.*

Theorem 6 *Let* (Λ, ρ) *be a metric space and* $(\mathbb{M}, \mathrm{d})$ *a complete metric space and let*

$$T : \Lambda \times \mathbb{M} \to \mathbb{M}$$

be a family of contraction mappings with uniform contraction constant k*, i.e.,*

$$\mathrm{d}\left(T(\lambda, x), T(\lambda, y)\right) \leq k\mathrm{d}(x, y),\ \forall \lambda \in \Lambda,\ \forall x, y \in \mathbb{M}.$$

Further more assume that for each $x \in \mathbb{M}$ *the mapping*

$$\lambda \mapsto T(\lambda, x)$$

is a continuous mapping from Λ *to* $\mathbb{M}$*. Then for each* $\lambda \in \Lambda$*,* $T(\lambda, \cdot)$ *has a unique fixed point* $x(\lambda) \in \mathbb{M}$*, and the mapping*

$$\lambda \mapsto x(\lambda),$$

is a continuous mapping from Λ *to* $\mathbb{M}$*.*

Theorem 7 *Let* $\mathbb{M}$ *be a Hilbert space and let*

$$T : \mathbb{M} \to \mathbb{M},$$

be a mapping such that for some constant $0 < \beta < 1$

$$\|T(u) - T(v)\| \leq \beta\|u - v\|, \quad \forall u, v \in \mathbb{M}.$$

Then for any $w \in \mathbb{M}$*, the equation*

$$u + T(u) = w \tag{5}$$

has a unique solution $u = u(w)$*, and the mapping* $w \mapsto u(w)$ *is continuous.*

Theorem 8 *Let* $\mathbb{M}$ *be a Hilbert space and let*

$$T : \mathbb{M} \to \mathbb{M},$$

be a monotone mapping

$$\mathrm{Re}\left((T(u) - T(v), u - v)\right) \geq 0, \quad \forall u, v \in \mathbb{M},$$

such that for some constant $\beta > 0$

$$\|T(u) - T(v)\| \leq \beta\|u - v\|, \quad \forall u, v \in \mathbb{M}.$$

Then for any $w \in \mathbb{M}$*, the equation*

$$u + T(u) = w \tag{6}$$

has a unique solution $u = u(w)$*, and the mapping* $w \mapsto u(w)$ *is continuous.*

Inverse Function Theorem: Prove Theorem 9.24 of Baby Rudin 3d Editor

Implicit Function Theorem: Prove Theorem 9.28 of Baby Rudin 3d Editor