Bifurcation Theory
Center Manifolds

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1. (Center Manifolds for Maps) Consider a C^k-smooth map
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Bx + f(x,y) \\ Cy + g(x,y) \end{pmatrix} \tag{1}$$
where $x \in \mathbb{R}^{n_c+n_u}$, $y \in \mathbb{R}^{n_s}$, and f and g have only non-linear terms. Suppose that the $(n_c+n_u)\times(n_c+n_u)$ matrix B has n_c eigenvalues with $|\lambda| = 1$ and n_u eigenvalues with $|\lambda| > 1$, while all n_s eigenvalues of the $n_s \times n_s$ matrix C satisfy $|\lambda| < 1$. Let $n = n_s + n_c + n_u$.

   Prove the following Theorem using the Graph Transform

   Center-Unstable Manifold Assume $f(0,0) = 0$, $g(0,0) = 0$, and that the functions f and g have sufficiently small bounds and sufficiently small Lipschitz bounds Lip(f),Lip(g) on $\mathbb{R}^n$. Then the map has an invariant manifold
$$W^{cu} = \{(x, h(x)) : x \in \mathbb{R}^{n_c+n_u}\}, \tag{2}$$
   where $h : \mathbb{R}^{n_c+n_u} \to \mathbb{R}^{n_s}$ is a bounded and Lipschitz map satisfying $h(0) = 0$.

2. Consider the map
$$\bar{x} = a - bx - x^3. \tag{3}$$
   (a) Find equations of the bifurcation curves for the fixed points of this map and draw these curves in the plane of parameters (a, b).
   (b) On the curve which corresponds to the existence of a fixed point with the multiplier equal to -1, and the points from which a bifurcation curve emanates which corresponds to a period 2 point with a multiplier $+1$.

3. Exercise 4.3 of Chicone

4. Show that the center Manifold is not unique in general. Why in the proof of Theorem 4.1 in Chicone, he claims there is a unique center manifold?

5. Consider the system
$$\dot{x} = f(x,r)$$
   satisying
$$f(0,0) = \partial_x f(0,0) = 0, \partial_r f(0,0) = 1 \quad \text{and} \quad \partial_{xx} f(0,0) = -2$$
   Prove that there exists a neighborhood of the origin, and there a smooth coordinate transformation close to the identity
$$(x,t) \mapsto (X,T) \tag{4}$$
   of the form $X = x\Phi(x)$ and $\frac{dT}{dt} = \Psi(x,r)$ such that the system is transformed into
$$\frac{dX}{dt} = r - X^2$$
   Classify the bifurcation. In fact, it is enough the assume that $\partial_r f(0,0) > 0$ and $\partial_{xx} f(0,0) < 0$, why?

6. Study bifurcations in the system

$$\begin{aligned} \dot{x} &= a - x - y^2 \\ \dot{y} &= b - y - 2xy \end{aligned}$$

as parameters a and b vary. Namely, do the following:

(a) Show that the system cannot have periodic orbits.

(b) Show that the only bifurcations of equilibria of this system correspond to a single zero eigenvalue of the linearisation matrix.

(c) Draw the bifurcation curve on the (a, b) plane.

(d) For each of the regions, into which this curve divides the (a,b) plane, determine the number of equilibria and their stability.

7. Formulate and prove a theorem based on the implicit function theorem that can be used to show that a small perturbation of a family of differential equations with a saddle-node bifurcation has a nearby saddle-node bifurcation. A possible formulation can have the following flavor: suppose $\dot{x} = f(x, \mu)$ with $\mu \in \mathbb{R}$ unfolds a saddle-node bifurcation. Consider a family $\dot{x} = f(x, \mu, \nu)$ with $f(x, \mu, 0) = f(x, \mu)$. Prove that $\dot{x} = f(x, \mu, \nu)$ unfolds a saddle-node for each small value of ν.

8. Determine bifurcations, stability of critical elements and sketch the bifurcation diagram of

$$\ddot{y} + \dot{y} + y(2y^2 + \lambda(\lambda - 2)) = 0,$$

9. Find center manifolds to second order and reduced systems to third order for the following problems:

(a) $\dot{x} = x^2 y + \alpha z^2 2, \dot{y} = -y + x^2 + zy, \dot{z} = z - y^2 + xy.$

(b) $(x, y, z) \mapsto (-x + yz, -\frac{1}{2}y + x^2, 2z - xy).$