Solução Geral

DADO $\vec{x} \in \mathbb{R}^n$ & $A \in \mathbb{R}^{n \times n}$

Considere o P.V.I.

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad , \quad \vec{x}(0) = x_0$$

Então a solução é única dada

$$\vec{x}(t) = e^{At}\vec{x}_0$$

ONDE

$$\vec{x}(t) = \underbrace{\left[\mathbb{1} + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots\right]}_{\text{EXPONENCIAL DE MATRIZ}}\vec{x}_0$$

Prova:

$$\left(\vec{x}(t)\right)' = \left(\mathbb{1} + At + \frac{1}{2}A^2t^2 + \dots\right)'\vec{x}_0$$

$$= \left(A + A^2t + A^3\frac{t^2}{2} + \dots\right)\vec{x}_0$$

$$= A\underbrace{\left(\mathbb{1} + At + \frac{A^2t^2}{2} + \dots\right)}_{e^{At}}\vec{x}_0$$

$$= A\vec{x}(t)$$

PELO TEOREMA DE PICARD A SOLUÇÃO É ÚNICA

NOTA: SE $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

$$e^A \neq \begin{pmatrix} e^a & e^b \\ e^0 & e^c \end{pmatrix}$$

Thm: $A \in \mathbb{R}^{n \times n}$

$$A\vec{v} = \lambda\vec{v} \iff e^{A}\vec{v} = e^{\lambda}\vec{v}$$

Prova: $\Rightarrow$

$$e^{A}\vec{v} = \left(\mathbb{1} + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots\right)\vec{v}$$

NOTE $A^2\vec{v} = \lambda^2\vec{v}$ & $A^n\vec{v} = \lambda^n\vec{v}$

$$e^{A}\vec{v} = \left(\vec{v} + A\vec{v} + \frac{1}{2}A^2\vec{v} + \cdots\right)$$

$$= \left(\vec{v} + \lambda\vec{v} + \frac{1}{2}\lambda^2\vec{v} + \cdots\right)$$

$$= \left(1 + \lambda + \frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3 + \dots\right)\vec{v}$$

$$= e^{\lambda}\vec{v}$$

CASO 2×2 $A \in \mathbb{R}^{2\times 2}$

$$A\vec{u} = \lambda\vec{u}$$
$$A\vec{v} = \beta\vec{v} \qquad \lambda \neq \beta$$

Prop: $\Gamma = \{\vec{u}, \vec{v}\}$ é base de $\mathbb{R}^2$

PORTANTO $\forall\, \vec{x}_0 \in \mathbb{R}^2$

$$\vec{x}_0 = c_1\vec{u} + c_2\,\cancel{\vec{u}_2}\,\vec{v}$$

$$\vec{x}(t) = e^{At}\vec{x}_0 = e^{At}\left(c_1\vec{u} + c_2\vec{v}\right)$$

$$= c_1 e^{At}\vec{u} + c_2 e^{At}\vec{v}$$

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{u} + c_2 e^{\beta t}\vec{v}$$

Ex

$$A = \begin{pmatrix} -1 & -4 \\ -5 & -2 \end{pmatrix}$$

Pol carac $\quad p(\lambda) = \det(A - \lambda \mathbb{1})$

$$p(\lambda) = \begin{vmatrix} -1-\lambda & -4 \\ -5 & -2-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 18$$

$$p(\lambda) = 0 \Rightarrow \begin{array}{l} \lambda_1 = -6 \\ \lambda_2 = 3 \end{array}$$

AUTOVALORES $\lambda_1 = -6$

AUTOVETOR $\quad v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$$[A - (-6)\mathbb{1}]\, v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -4 \\ -5 & +4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} -5a + 4b = 0 \\ b = \dfrac{5}{4} a \end{array}$$

$$v_1 = \begin{pmatrix} 1 \\ 5/4 \end{pmatrix}$$

AUTOVALOR $\lambda_2 = 3$

AUTOVETOR $\quad v_2 = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} -4 & -4 \\ -5 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4a + 4b = 0$$

$$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

__NOTE:__ $\Gamma = \left\{ \begin{pmatrix} 1 \\ 5/4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ É BASE DE $\mathbb{R}^2$.

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2 \qquad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 5/4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = c_1 A \begin{pmatrix} 1 \\ 5/4 \end{pmatrix} + c_2 A \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= -6 c_1 \begin{pmatrix} 1 \\ 5/4 \end{pmatrix} + 3 c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \dot{\vec{x}} = \underbrace{\begin{pmatrix} -1 & -4 \\ -5 & -2 \end{pmatrix}}_{A} \vec{x} \\ \vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}$$

Como $A\begin{pmatrix} 1 \\ 5/4 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 5/4 \end{pmatrix}$

& $A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

<u>TEMOS</u>

$$\vec{x}(t) = c_1 e^{-6t}\begin{pmatrix} 1 \\ 5/4 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 5/4 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\Rightarrow$ $c_1 = 0$ & $c_2 = 1$.

EDO 2º ORDEM ESCALAR COMO SISTEMA

$x : \mathbb{R} \to \mathbb{R}$:

EQ CARA

$$x'' + a x' + b x = 0 \Rightarrow \lambda^2 + a\lambda + b = 0$$

<u>1º PASSO</u> | $y = x' \Rightarrow x'' = y'$

$\therefore\ y' = -a \underbrace{x'}_{y} - bx$

<u>2º PASSO</u> |

$$\begin{aligned} x' &= y \\ y' &= -ay - bx \end{aligned} \quad ; \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{\vec{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}}_{A} \vec{x}$$

$$p_A(\lambda) = \det(A - \lambda \mathbb{1}) = \begin{vmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{vmatrix}$$

$$= \lambda^2 + a\lambda + b \ \blacksquare$$