Mesonic Eightfold Way From Dynamics and Confinement in Strongly Coupled Lattice QCD

Antônio Francisco Neto*
Núcleo de Física, Campus Prof. Alberto Ceralho - UFS
49500-000 Itabaiana SE, Brazil

Michael O’Carroll and Paulo A. Faria da Veiga†
Departamento de Matemática Aplicada e Estatística, ICMC-USP
C.P. 668, 13560-970 São Carlos SP, Brazil

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We consider a 3 + 1 lattice QCD model with three quark flavors, local SU(3) gauge symmetry, global SU(3)$_J$ isospin or flavor symmetry, in an imaginary-time formulation and with strong coupling (a small hopping parameter $\kappa > 0$ and a plaquette coupling $\beta > 0$, $0 < \beta \ll \kappa \ll 1$). Associated with the model there is an underlying physical quantum mechanical Hilbert space $\mathcal{H}$ with positive self-adjoint energy and self-adjoint momentum operators, which, via a Feynman-Kac formula, enables us to introduce spectral representations for correlations and obtain the low-lying energy-momentum spectrum exactly. Using the decoupling of hyperplane method and concentrating on the subspace $\mathcal{H}_e \subset \mathcal{H}$ of vectors with an even number of quarks, we obtain the one-particle spectrum showing the existence of 36 meson states from dynamical first principles, i.e., directly from the quark-gluon dynamics. Besides the SU(3)$_J$ quantum numbers (total hypercharge, quadratic Casimir $C_2$, total isospin and its 3rd component), the basic excitations also carry spin labels. The total spin operator $J$ and its z-component $J_z$ are defined using the SU(3) decomposition into a SU(3) and one nonet associated with the pseudo-scalar mesons ($\mathcal{O}(C_2 = 0)$) and octet ($\mathcal{O}(C_2 = 3)$). The particles are detected by isolated dispersion curves $w(\vec{p})$ in the energy-momentum spectrum. They are all of the form, for $\beta = 0$, $w(\vec{p}) = -2 \ln \kappa - 3 \kappa^2/2 + (1/4) \kappa^4 \sum_{j=1}^3 2(1 - \cos p^j) + \kappa^4 r(\kappa, \vec{p})$, with $|r(\kappa, \vec{p})| \leq \text{const}$. The meson masses are given by $m(\kappa) = -2 \ln \kappa - 3 \kappa^2/2 + \kappa^4 r(\kappa)$, with $r(0) \neq 0$ and $r(\kappa)$ real analytic; for $\beta \neq 0$, $m(\kappa) + 2 \ln \kappa$ is jointly analytic in $\kappa$ and $\beta$. For a fixed nonet, the mass of the vector mesons are independent of $J_z$ and are all equal within each octet. All singlet masses are also equal for the vector mesons. For $\beta = 0$, up to and including $O(\kappa^6)$, for each nonet, the masses of the octet and the singlet are found to be equal. All members of each octet have identical dispersions. Other dispersion curves may differ. Indeed, there is a pseudo-scalar, vector meson mass splitting (between $J = 0$ and $J = 1$) given by $2\kappa^4 + O(\kappa^8)$. The splitting persists for $\beta \neq 0$. Using a correlation subtraction method, we show the 36 meson states give the only spectrum in $\mathcal{H}_e$ up to near the two-meson threshold of $\approx -4 \ln \kappa$. Combining our present result with a previous one for baryons shows that the model exhibits confinement up to near the two-meson threshold.

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I. INTRODUCTION

A landmark in particle physics was achieved when Gell-Mann and Ne’eman independently proposed the eightfold way scheme to classify the then known hadronic particles (see Refs. [1–5]). This model asserts that every hadron is composed of quarks with three flavors up ($u$), down ($d$) and strange ($s$). The baryons are made of three quarks and the mesons are made of a pair quark-antiquark. Later, a color dynamics was introduced for the quarks via an exchange of gauge vector bosons called gluons and today the local gauge model of interacting quarks and gluons, based on the color symmetry SU(3)$_c$, known as Quantum Chromodynamics (QCD), is the best candidate to describe the strong

*Electronic address: afneto@fisica.ufs.br
†Electronic address: veiga@icmc.usp.br
interactions.

The lattice regularization of the continuum theory was introduced by Wilson in Ref. [6]. Precisely, in Ref. [6], an imaginary-time functional integral formulation for lattice QCD was developed. In this framework, there are basically two ingredients: quarks, obeying Fermi-Dirac statistics, carrying flavor, color and spin labels and located in each lattice site; colored string bits connecting adjacent points on the lattice. Besides preserving gauge invariance and being free of ultraviolet singularities, the lattice formulation is powerful enough to e.g. obtain the first results on the QCD particle spectrum and to exhibit confinement, i.e. isolated quarks are not observed. Within this construction, one formally recovers the continuum theory in the scaling limit (i.e. the lattice spacing going to zero). Later, it was shown by Osterwalder and Seiler that the lattice regularization of Wilson has the property of reflection positivity (see Refs. [7, 8] for more details). This property enables the construction of the quantum mechanical Hilbert space of physical states $\mathcal{H}$ and allows us to define a positive self-adjoint energy and self-adjoint momentum operators. Within this framework, a Feynman-Kac formula is established and the energy-momentum ($E$-M) spectrum can be investigated.

The low-lying E-M spectrum (one-particle and two-particle bound states) was rigorously determined exactly in Refs. [9–19] for increasingly complex SU(3)$_c$ lattice QCD models with one and two flavors in the strong coupling regime, i.e. with the hopping parameter $\kappa$ and plaquette coupling $\beta$ satisfying $0 < \beta < \kappa < 1$. In this regime, more recently, the SU(3)$_f$ scheme for baryons was validated in Refs. [20, 21] for the 3 + 1-dimensional lattice QCD with three quark flavors and using the Wilson action. 56 eightfold way baryon states of mass $\approx -3\ln\kappa$ were obtained from first principles, i.e. directly from the quark-gluon dynamics. Concentrating on the subspace $\mathcal{H}_o \subset \mathcal{H}$ of vectors with an odd number of quarks, and applying a correlation subtraction method, the eightfold way baryon and anti-baryon spectrum was shown to be the only spectrum up to near the meson-baryon energy threshold of $\approx -5\ln\kappa$. More precisely, up to near the meson-baryon energy threshold, all the E-M spectrum is generated by the 56 eightfold way baryon fields, and the baryon particles are strongly bound bound states with three quarks. For each baryon, there is a corresponding anti-baryon, related by charge conjugation and with identical spectral properties. The reason for the restriction $\beta << \kappa$ is that in this region of parameters the hadron spectrum (mesons and baryons of asymptotic masses $-2\ln\kappa$ and $-3\ln\kappa$, respectively) is the low-lying spectrum. If, on the other hand, $\beta >> \kappa$ then the low-lying spectrum consists of only glueballs (of asymptotic mass $-4\ln\beta$) and their excitations (see Ref. [22]).

The baryons, are detected as complex momentum singularities of the Fourier transform of a two-baryon correlation, and are rigorously related to the E-M spectrum via a spectral representation to this quantity. The spectral representation for the two-point correlation is derived using a Feynman-Kac formula. The 56 baryon states can be grouped into two octets of total spin $J = 1/2$ ($J_z = \pm 1/2$), and four decuplets of total spin $J = 3/2$ ($J_z = \pm 1/2, \pm 3/2$). On the lattice, there is only the discrete $\pi/2$-rotation group. However, we can adapt the treatment usually employed in solid state physics and use the structure of point groups (see Ref. [23]) to rigorously introduce total spin operators $J$ and spin z-component $J_z$ so that, for zero momentum states, there is a partial restoration of rotational symmetry meaning that these operators inherit the same structure as for the continuum. Using this, shows that the masses within the octets and within the decuplets are independent of $J_z$. However, there is a mass splitting between the octets and the decuplets given, to leading order, of $3e^\beta/4$ at $\beta = 0$. For $\beta \neq 0$, for $m_b(\kappa)$ the baryon mass, $m_b(\kappa) + 3\ln\kappa$ is jointly analytic in $\kappa$ and $\beta$. In particular the mass splitting between octet and decuplet persists for $\beta \neq 0$.

It is worth remarking that the work of Refs. [20, 21] is not the first publication on the one-particle spectrum of lattice QCD models with three flavors. Specially on the 80’s, many papers were devoted to the existence of baryons, namely, e.g. the work of Refs. [24, 25]. However, these papers do not rely on spectral representations. This can be problematic when tiny splitting among the states are present. Also, the determination of momentum singularities of the two-point function via the zeros of uncontrolled expansion in the denominator of the Fourier transform of approximate propagators leaves unanswered the question of the nature and the existence of the supposed singularity. The same kind of problems may show up in works where the masses are determined by the exponential decay rates of two-point functions, as what is usually done in numerical simulations. We remark that we work with the exact correlation function, its convolution inverse and their Fourier transforms.

In this work, in the strong coupling regime, and using the decoupling of hyperplane method (see Ref. [26–29]), we complete the exact determination of the one-particle E-M spectrum initiated in Refs. [20, 21]. For a pedagogical presentation of the basic principles of the hyperplane decoupling method see Ref. [28]. Here, we consider the even subspace $\mathcal{H}_e \subset \mathcal{H}$ and show the existence of the 36 eightfold way meson (of asymptotic mass $-2\ln\kappa$) states.

The hyperplane decoupling method has many nice features:

- It enables us to obtain the basic local gauge invariant excitation fields without any a priori guesswork. As will be shown, linear combination of these fields can be identified with the eightfold way particles, namely, the pseudo-scalar and vector mesons.

- It gives good control of the global decay properties of the correlation functions involved.
• It enables us to show that the spectrum is generated by isolated dispersion curves, i.e. the upper gap property.

• It permits us to show that the only spectrum in all $\mathcal{H}_c$ is generated by the eightfold way particles.

Using a meson correlation subtraction method, we also show that the spectrum generated by the 36 eightfold way meson states, bound states of a quark-antiquark, is the only spectrum in the whole $\mathcal{H}_c$, up to near the two-meson threshold of $\approx -4\ln \kappa$. These 36 states can be grouped into four SU(3)$_f$ nonets; one associated with the pseudo-scalar mesons ($J = 0$) and three with the vector mesons ($J = 1, J_z = 0, \pm 1$). Each nonet admits a further decomposition into a singlet (with quadratic SU(3)$_f$ Casimir $C_2 = 0$) and an octet ($C_2 = 3$). The 36 mesons are labelled (and distinguished by this labelling) by $J, J_z, C_2$, and the quantum numbers of total isospin $I$ (its square), third-component of total isospin $I_3$ and total hypercharge $Y$.

The meson particles are detected by isolated dispersion curves $w(\vec{p})$ in the energy-momentum spectrum. They are of the form, for $\beta = 0$, $w(\vec{p}) = -2 \ln \kappa - 3\kappa^2/2 + (1/4)\kappa^2 \sum_{j=1}^3 2(1 - \cos p^j) + \kappa^4 r(\kappa, \vec{p})$, with $|r(\kappa, \vec{p})| \leq \text{const}$. For the pseudo-scalar mesons $r(\kappa, \vec{p})$ is jointly analytic in $\kappa$ and $p^j$, for $|\kappa|$ and $|\text{Im} p^j|$ small. The meson masses are given by $m(\kappa) = -2 \ln \kappa - 3\kappa^2/2 + \kappa^4 r(\kappa)$, with $r(0) \neq 0$ and $r(\kappa)$ real analytic. The non-singular part of the mass, i.e. $r(\kappa)$, is jointly analytic in $\kappa$ and $\beta$. For a fixed nonet, the masses of all vector mesons are independent of $J_z$ and are all equal within each octet. All singlet masses are also equal. For $\beta = 0$, up to and including $O(\kappa^4)$, each nonet, the masses of the octet and the singlet are found to be equal. All members of each octet have identical dispersions. Other dispersion curves may differ. Indeed, there is a pseudo-scalar, vector meson mass splitting (between $J = 0$ and $J = 1$) given by $2\kappa^4 + O(\kappa^6)$, and, by analyticity, the splitting persists for $\beta \neq 0$. Using a correlation subtraction method, we show the 36 meson states give the only spectrum in $\mathcal{H}_c$ up to near the two-meson threshold of $\approx -4\ln \kappa$.

Combining the present result with the results of Refs. [20, 21] shows confinement up to the two-meson threshold. We stress that, even within the limitation of dealing with only three quark flavors, since not all the eightfold way meson states, bound states of a quark-antiquark, is the only spectrum in the whole $\mathcal{H}_c$.

II. MODEL AND RESULTS

In the first subsection, we introduce the lattice QCD model, the physical quantum mechanical Hilbert space $\mathcal{H}$. In particular, the Feynman-Kac formula and the self-adjoint E-M operators are considered. We also state our main results on the E-M spectrum and the existence of meson particles in all $\mathcal{H}_c$ in Theorem 1. In the second subsection, the decoupling of hyperplane method is introduced and used to obtain the basic excitation states appearing in the two-
point function. To determine E-M spectrum we need the long distance and short distance behavior of the two-point function and its convolution inverse, which are presented in Theorems 2 and 3.

A. Model and the physical quantum mechanical Hilbert space

We use the same model presented in Refs. [20, 21]. The model is the SU(3)\(_f\) lattice QCD model with the partition function given formally by \( Z = \int e^{-S(\bar{\psi}, \psi, g)} d\bar{\psi} d\psi d\mu(g) \), and for a function \( F(\psi, \bar{\psi}, g) \), the normalized correlations are denoted by

\[
\langle F \rangle = \frac{1}{Z} \int F(\psi, \bar{\psi}, g) e^{-S(\bar{\psi}, \psi, g)} d\bar{\psi} d\psi d\mu(g).
\]

The action \( S = S(\psi, \bar{\psi}, g) \) is given by

\[
S(\psi, \bar{\psi}, g) = \frac{\kappa}{2} \sum_{u, \alpha, f} \bar{\psi}_{\alpha, \alpha, f}(u) \Gamma_{\alpha, \beta, f}^\rho(\psi_{\beta, \alpha, f}(u + e \rho) - \psi_{\beta, \alpha, f}(u)) + \sum_{u \in Z^d_\mathbb{Z}} \bar{\psi}_{\alpha, \alpha, f}(u) M_{\alpha, \beta} \psi_{\alpha, \beta, f}(u) - \frac{1}{g_0^2} \sum_p \chi(g_p),
\]

where the first sum is over \( u \in Z^d_\mathbb{Z}, \epsilon = \pm 1, \rho = 0, 1, 2, 3 \) and over repeated indices. Calling 0 the temporal direction, the lattice is given by \( Z^d_\mathbb{Z}, \) where \( u = (\nu^0, \vec{u}) = (\nu^0, U^2, U^3) \in Z^d_\mathbb{Z} \equiv \mathbb{Z}_1/2 \times \mathbb{Z}^3, \) where \( \mathbb{Z}_1/2 = \{ \pm 1/2, \pm 3/2, \ldots \}. \) For each site \( u \in Z^d_\mathbb{Z}, \) there are fermionic fields, represented by Grassmann variables, \( \psi_{\alpha, \alpha, f}(u), \) associated with a quark, and \( \bar{\psi}_{\alpha, \alpha, f}(u), \) to an anti-quark, which carry a Dirac spin \( \alpha = 1, 2, 3, \) a color \( a = 1, 2, 3 \) and flavor or “isospin” \( f = u, d, s = 1, 2, 3 \) index. \( \Gamma \) is related to the Dirac matrices by \( \Gamma^{\pm, e^\rho} = -1 \pm e^\rho. \) The \( \gamma^j \) are the Dirac 4 \times 4 matrices

\[
\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \quad \text{and} \quad \gamma^j = \begin{pmatrix} 0 & i\sigma^j \\ -i\sigma^j & 0 \end{pmatrix};
\]

\( \sigma^j, j = 1, 2, 3, \) denote the Pauli 2 \times 2 matrices and \( I_2 \) is the 2 \times 2 identity matrix. For each orientation bound on the lattice \( < u, u, \epsilon e^\rho > \) there is a matrix \( U(g_{u, u, \epsilon e^\rho}) \in SU(3)\) parameterized by the gauge group element \( U(g_{u, u, \epsilon e^\rho}) \) and satisfying \( U(g_{u, u, \epsilon e^\rho})^{-1} = U(g_{u, u, \epsilon e^\rho, u}). \) The parameters \( \kappa \) is the hopping parameter and \( \beta \equiv 1/g_0^2 \) is the plaquette coupling. The measure \( d\mu(g) \) is the product measure over non-oriented bonds of normalized \( SU(3). \) Haar measures (see Ref. [31]). There is only one integration variable per bond, so that \( g_{uv} \) and \( g_{vu}^{-1} \) are not treated as distinct integration variables. The integrals over Grassmann fields are defined according to Ref. [32]. For a polynomial in the Grassmann variables with coefficients depending on the gauge variables, the fermionic integral is defined as the coefficient of the monomial of maximum degree, i.e. of \( \prod_{u, k} \psi_k(u) \bar{\psi}_k(u), \) \( k \equiv (a, \alpha, f). \) In Eq. (1), \( d\bar{\psi} d\psi \) means \( \prod_{u, k} d\psi_k(u) d\bar{\psi}_k(u) \) such that, with a normalization \( N = \{1\}, \) we have \( \langle \psi_k(x) \bar{\psi}_{k}(y) \rangle = (1/N) \int \psi_k(x) \bar{\psi}_{k}(y) e^{-\sum_{u, k, k_1} \psi_k(u) \bar{\psi}_{k_1}(u) O_{k, k_1} \psi_{k_1}(u) d\bar{\psi}_k} = O^{-1}_{\alpha_1 \alpha_2, \delta_{j_1 j_2}, \delta_{f_1 f_2}, \delta(x - y)} \), with a Kronecker delta for space-time coordinates, and where \( O \) is diagonal in the color and isospin indices.

By polymer expansion methods (see Refs. [7, 27, 28]), the thermodynamic limit of correlations exists and truncated correlations have exponential tree decay. The limiting correlation functions are lattice translational invariant. Furthermore, the correlation functions extend to analytic functions in the global coupling parameters \( \kappa \) and \( \beta = 1/(2g_0^2) \) and also in any finite number of local coupling parameters. For the formal hopping parameter expansion, see Refs. [33–35].

Associated with the SU(3)\(_f\) model, there is an underlying physical Hilbert space which we denote by \( \mathcal{H}. \) Starting from gauge invariant correlations, with support restricted to \( u^0 = 1/2 \) and letting \( T_0, \) \( T_i^x, i = 1, 2, 3, \) denoting translation of the functions of Grassmann and gauge variables (\( \gamma \) is used to denote Hilbert space operators) by \( x^0 \geq 0, \) \( \vec{x} = (x^1, x^2, x^3) \in \mathbb{Z}^3 \) there is the Feynman-Kac formula, for \( F \) and \( G \) only depending on coordinates with \( u^0 = 1/2, \) giving by

\[
(G, T_0, T_1^x, T_2^x, T_3^x, T^3 F)_{\mathcal{H}} = \langle [T_0^x T^3 F]_\Theta G \rangle
\]

where \( T^x = T_1^x T_2^x T_3^x \) and \( \Theta \) is an anti-linear operator which involves time reflection. Following Ref. [10], the action of \( \Theta \) on single fields is given by

\[
\Theta \psi_{\alpha, \alpha, f}(u) = (\gamma^0)_{\alpha \beta} \psi_{\alpha, \beta, f}(u)
\]

\[
\Theta \bar{\psi}_{\alpha, \alpha, f}(u) = \bar{\psi}_{\alpha, \beta, f}(u) (\gamma^0)_{\alpha \beta};
\]
where $u_t = (-u^0, \bar{u})$, for $A$ and $B$ monomials, $\Theta(AB) = \Theta(B)\Theta(A)$, and for a function of the gauge fields $\Theta f(\{g_{uv}\}) = f^*(\{g_{uv}\})$, $u, v \in Z^3$ where $*$ means complex conjugate. $\Theta$ extends anti-linearly to the algebra. For simplicity, we do not distinguish between Grassmann, gauge variables and their associated Hilbert space vector spaces in our notation. As linear operators in $\mathcal{H}$, $\mathcal{T}_\rho \rho = 0, 1, 2, 3$, are mutually commuting; $\mathcal{T}_0$ is self-adjoint, with $-1 \leq \mathcal{T}_0 \leq 1$, and $\mathcal{T}_{j=1,2,3}$ are unitary. So, we write $\mathcal{T}_0 = e^{i\mathcal{P}_0}$ and $\mathcal{F} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is the self-adjoint momentum operator with spectral points $\mathcal{p} \in \mathbb{T}^3 = (-\pi, \pi]^3$. Since $\mathcal{T}_0^2 \geq 0$, the energy operator $\mathcal{H} > 0$ can be defined by $\mathcal{T}_0^2 = e^{-2\mathcal{H}}$. We call a point in the E-M spectrum associated with spatial momentum $\mathcal{p} = 0$ a mass and, to be used below, we let $\mathcal{E}(\lambda_0, \bar{\lambda})$ be the product of the spectral families of $\mathcal{T}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. By the spectral theorem (see Ref. [36]), we have

$$\mathcal{T}_0 = \int_{-\pi}^{\pi} d\lambda^0 E_0(\lambda^0) \mathcal{T}_{j=1,2,3} = \int_{-\pi}^{\pi} e^{i\lambda_j} d\mathcal{F}_j(\lambda^j);$$

so that $\mathcal{E}(\lambda_0, \bar{\lambda}) = E_0(\lambda_0) \prod_{j=1}^3 \mathcal{F}_j(\lambda^j)$. The positivity condition $\langle \mathcal{F} \mathcal{F} \rangle \geq 0$ is established in Ref. [8], but there may be nonzero $F$’s such that $\langle \mathcal{F} \mathcal{F} \rangle = 0$. If the collection of such $F$’s is denoted by $\mathcal{N}$, a pre-Hilbert space $\mathcal{H}'$ can be constructed from the inner product $\langle \mathcal{G} \mathcal{F} \rangle$ and the physical Hilbert space $\mathcal{H}$ is the completion of the quotient space $\mathcal{H}'/\mathcal{N}$, including also the cartesian product of the inner space sectors, the color space $\mathbb{C}^3$, the spin space $\mathbb{C}^4$ and the isospin space $\mathbb{C}^3$.

Concerning the parameters $\kappa$, $\beta$ it is to be understood that the following conditions hold in the sequel: There exist $\kappa_0 > 0$, $\beta_0 > 0$ and $\beta_0/\kappa_0 > 0$ sufficiently small. Under this condition our results hold for all physical values of $\kappa$ and $\beta$ such that $\kappa < \kappa_0$, $\beta < \beta_0$ and $\beta/\kappa < \beta_0/\kappa_0 << 1$. The main result of this paper is summarized in the Theorem 1 below:

**Theorem 1** The low-lying energy-momentum spectrum of the lattice QCD model given by the action of Eq. (2), in the strong coupling regime, in the even subspace $\mathcal{H}_e \subset \mathcal{H}$ and up to near the two-meson threshold of $\approx -4\ln \kappa$ is generated by 36 states, bound states of a quark-antiquark. These 36 are labelled by the $SU(3)_f$ quantum numbers $I, I_3, Y$ and $C_2$. Also, for zero momentum states, a spin labelling can be introduced. The meson states can be distinguished and grouped into three $SU(3)_f$ nonets, associated with the vector mesons ($J = 1, I_3 = 0, \pm 1$), and one nonet associated with the pseudo-scalar mesons ($J = 0$). Each nonet admits a further decomposition into a $SU(3)_f$ singlet ($C_2 = 0$) and an octet ($C_2 = 3$). The particles are detected by isolated dispersion curves $w(\mathcal{p})$ in the energy-momentum spectrum. The 36 dispersion curves are all of the form, for $\beta = 0$,

$$w(\mathcal{p}) = -2\ln \kappa - 3\beta^2/2 + (1/4)\kappa^2 \sum_{j=1}^3 2(1 - \cos p_j) + \kappa^4 r(\kappa, \mathcal{p}) ,$$

with $|r(\kappa, \mathcal{p})| \leq \text{const.}$ For the pseudo-scalar mesons, we can show that $r(\kappa, \mathcal{p})$ is jointly analytic in $\kappa$ and $p^j$, for $|\kappa|$ and $|\text{Im} p^j|$ small. The meson masses are of the form

$$m(\kappa) = -2\ln \kappa - 3\beta^2/2 + \kappa^4 r(\kappa) ,$$

with $r(0) \neq 0$ and $r(\kappa)$ real analytic. The $m(\kappa) + 2\ln \kappa$ is jointly analytic in $\kappa$ and $\beta$. For a fixed nonet, the masses of all vector mesons are independent of $J_3$ and are all equal within each octet. All singlet masses are also equal for the vector mesons. For $\beta = 0$, up to and including $O(\kappa^4)$, for each nonet, the masses of the octet and the singlet are found to be equal. All members of each octet have identical dispersion curves. Other dispersion curves may differ. Indeed, there is a pseudo-scalar, vector meson mass splitting (between $J = 0$ and $J = 1$) given by $2\kappa^4 + O(\kappa^6)$; the splitting persists for $\beta \neq 0$. Up to near the two-meson threshold of $\approx -4\ln \kappa$ the one-hadron spectrum in the physical quantum mechanical Hilbert space $\mathcal{H}$ of the QCD lattice model under consideration is only given by the eightfold way gauge-invariant baryon and meson states and confinement is thus proven up to this threshold. Last, combining the above results with similar results for the eightfold way baryons, i.e. the eightfold way baryons are the only spectrum in all $\mathcal{H}_e$ up to near the meson-baryon threshold ($\approx -5\ln \kappa$), we also show that, up to near the two-meson threshold, the one-hadron spectrum in $\mathcal{H}$ of our lattice QCD model consists solely of the eightfold way gauge-invariant baryon and meson states and confinement is thus proven up to near this threshold.

**Proof:** The proof of Theorem 1 follows from Theorems 2 and 3 stating global bounds and short distance behavior in $\kappa$, respectively, of the two-point function and its convolution inverse. The fact that the masses depend only on the total spin follows from the analysis of Section III B. On Subsection III D, using Theorems 2 and 3, the pseudo-scalar and vector mesons masses, dispersion curves and their multiplicities are determined exactly. Using a meson correlation subtraction method, in Section V, we extend our spectral results to all $\mathcal{H}_e$. 

$\square$
B. One-meson spectrum

We start by introducing a spectral representation for the two-point subtracted correlation. To obtain the spectral representation we use the F-K formula and observe that, for $M, L \in \mathcal{H}_e$ and with support at time $u^0 = 1/2$, denoting them by $M(1/2) \equiv M(1/2, \vec{0})$ and $L(1/2) \equiv L(1/2, \vec{0})$

$$\left( 1 - P_{\vec{0}} \right) M(1/2), T_{\vec{0}}^{[u^0-u^0]}-1 \tilde{T}_{\vec{0}}^{-\vec{u}} (1 - P_{\vec{0}}) L(1/2) \right)_{\mathcal{H}} = \langle [T_{\vec{0}}^{[u^0-u^0]}-1 \tilde{T}_{\vec{0}}^{-\vec{u}} L(1/2)](\Theta M)(-1/2) \rangle_T. \quad (4)$$

Note that $P_{\vec{0}}$ is the projection onto the vacuum state $\Omega \equiv 1$, since we are interested in the spectrum generated by vectors orthogonal to the vacuum. From the relation obtained above we have for $v^0 > u^0$

$$\langle [T_{\vec{0}}^{[u^0-v^0]}-1 \tilde{T}_{\vec{0}}^{-\vec{u}} L(1/2)](\Theta M)(-1/2) \rangle_T = \langle [T_{\vec{0}}^{[u^0-v^0]}-1 \tilde{T}_{\vec{0}}^{-\vec{u}} L(1/2)](\Theta M)(-1/2) \rangle_T$$

where we have defined $L(v^0, \vec{v}) \equiv T_{\vec{0}}^{[u^0-v^0]}-1 \tilde{T}_{\vec{0}}^{-\vec{v}} L(1/2)$ and $\Theta M(u^0, \vec{u}) = T_{\vec{0}}^{[u^0-u^0]}-1 \tilde{T}_{\vec{0}}^{-\vec{u}} (\Theta M)(-1/2)$. Similarly, for $v^0 < u^0$, we have, moving the energy and momentum operators to the LHS and taking the complex conjugate,

$$\left( 1 - P_{\vec{0}} \right) M(1/2), T_{\vec{0}}^{[u^0-v^0]}-1 \tilde{T}_{\vec{0}}^{-\vec{u}} (1 - P_{\vec{0}}) M(1/2) \right)^* = \langle [T_{\vec{0}}^{[u^0-v^0]}-1 \tilde{T}_{\vec{0}}^{-\vec{u}} M(1/2)](\Theta L)(-1/2) \rangle_T. \quad (5)$$

From the relation obtained above we have $\langle [T_{\vec{0}}^{[u^0-v^0]}-1 \tilde{T}_{\vec{0}}^{-\vec{u}} M(1/2, \vec{u})]\Theta L(-1/2, \vec{v}) \rangle_T^* = \langle M(u^0, \vec{u}) \Theta L(v^0, \vec{v}) \rangle_T^*.$

With this we define the general two-point function below, with $M, L \in \mathcal{H}_e$, and using the $u^0 < v^0$ value to extend the correlation values to $u^0 = v^0$,

$$G_{ML}(u, v) \equiv \begin{cases} \langle \Theta M(u) L(v) \rangle_T, & u^0 \leq v^0 \\ \langle M(u) \Theta L(v) \rangle_T, & u^0 > v^0 \end{cases}. \quad (6)$$

The correlation in Eq. (6) admits the spectral representation, for $x^0 \neq 0, x = (x^0, \vec{x}) = v - u, G_{ML}(x) = G_{ML}(0, v - u) = G_{ML}(u, v)$,

$$G_{ML}(x) = \int_{-1}^{1} \int_{\mathcal{T}} \lambda |x|^n \cdot 1 \cdot e^{i \lambda \vec{x}} d(1 - P_{\vec{0}}) M(1/2), \mathcal{E}(\lambda_0, \vec{\lambda})(1 - P_{\vec{0}}) L(1/2) \mathcal{H}.$$ 

Our starting point to obtain the eightfold way mesons, as mentioned before, is the product structure obtained using the decoupling of hyperplane method. Before we introduce the method we use a convenient form to represent the subtracted two-point function, namely, the duplicate of variable representation. By this we mean to replace the numerator $N$ and denominator $D$ of the general two-point function of Eq. (6) and we use the notation $N^{(m,n)}(u, v) = [D^{(m,n)}(u, v)],$ with $m, n \geq 0$, meaning the coefficient of $\kappa^m \beta^p_{n}$ in the expansion of the numerator [denominator]. First, $\mathcal{N}^{(0,0)} = 0$ since $M(u)$ decouples from $L(v)$ under the expectation $\langle \cdot \rangle$ in the two-point subtracted function. For $\kappa^0_{n}$ the first non-vanishing coefficient is $\beta^p_{n}$, arising from four vertical plaquettes composing the four vertical sides of a cube. Next, $\mathcal{N}^{(2m+1, n)} = 0 = D^{(2m+1, n)},$ recalling that each expectation factorizes and each factor has an odd number of
fermion fields giving zero. The term $\mathcal{N}^{(0,1)} = 0 = \mathcal{D}^{(0,1)}$ is trivially zero due to gauge integration. The term $\mathcal{N}^{(0,2)}$, corresponding to two superposed vertical plaquettes with opposite orientation, is zero. More precisely, using the gauge integral $\mathcal{I}_2$ [see Eq. (B2)] and the fact that, as in $\mathcal{N}^{(0,0)} = 0$, the fields decouple we get the desired result. $\mathcal{N}^{(0,3)}$ is zero, corresponding to three superposed vertical plaquettes with the same orientation, and using the gauge integral $\mathcal{I}_3$ [see Eq. (B3)] for the vertical sides of the superposed plaquettes. Therefore, collecting our results for $m + n \leq 3$ we get:

$$\mathcal{G}_{ML}(u,v) = \mathcal{G}_{ML}^{(2,0)}(u,v)\kappa_p \beta_p + \text{(higher order terms)}$$

where we have used

$$\mathcal{G}_{ML}^{(0,0)}(u,v) = \mathcal{G}_{ML}^{(1,0)}(u,v) = \mathcal{G}_{ML}^{(0,1)}(u,v) = \mathcal{G}_{ML}^{(1,1)}(u,v) = \mathcal{G}_{ML}^{(0,2)}(u,v) = \mathcal{G}_{ML}^{(0,3)}(u,v) = \mathcal{G}_{ML}^{(1,2)}(u,v) = \mathcal{G}_{ML}^{(0,4)}(u,v) = 0.$$

Considering the second $\kappa_p$ derivative of $\mathcal{G}$, i.e. $\mathcal{G}_{ML}^{(2,0)}$, and for the time ordering $u^0 \leq p < v^0$

$$\langle (M(u)L(v))^{(2,0)} \rangle = \sum_{\bar{w}} \langle [\mathcal{M}(u) - M'(u)] \mathcal{M}_{\bar{g}}(p, \bar{w}) \rangle^{(0,0)} \langle \mathcal{M}_{\bar{g}}(p + 1, \bar{w}) [L(v) - L'(v)] \rangle^{(0,0)}$$

which we write schematically as

$$\mathcal{G}_{LL}^{(2,0)}(u,v) = \left[ \mathcal{G}_{LM}^{(0,0)} \circ \mathcal{G}_{ML}^{(0,0)} \right] (u,v).$$

Similarly, for the time ordering $u^0 > p \geq v^0$ we have, for $L = \Theta M$,

$$\langle (M(u)L(v))^{(2,0)} \rangle = \sum_{\bar{w}} \langle [\mathcal{M}(u) - M'(u)] \mathcal{M}_{\bar{g}}(p, \bar{w}) \rangle^{(0,0)} \langle \mathcal{M}_{\bar{g}}(p, \bar{w}) [L(v) - L'(v)] \rangle^{(0,0)}$$

written also as, after taking the complex conjugate of Eq. (11),

$$\mathcal{G}_{MM}^{(2,0)}(u,v) = \left[ \mathcal{G}_{MM}^{(0,0)} \circ \mathcal{G}_{MM}^{(0,0)} \right] (u,v).$$

In Eq. (10) we have defined

$$\mathcal{M}_{\bar{g}} = \frac{1}{\sqrt{3}} \bar{\psi}_{a, \gamma, g_1} \psi_{a, \gamma, g_2}$$

and similarly we define the $\mathcal{M}$ fields

$$\mathcal{M}_{\bar{g}} = \frac{1}{\sqrt{3}} \bar{\psi}_{a, \gamma, g_1} \psi_{a, \gamma, g_2},$$

i.e. making the change $\psi \rightarrow \bar{\psi}$ and $\bar{\psi} \rightarrow \psi$ and preserving the color ($a = 1, 2, 3$), spin ($\alpha_u, \beta_u, \gamma_u = 1, 2, 3$), and flavor or isospin index ($f, g, h = u, d, s = 1, 2, 3$). In Eqs. (12), (13) and in the sequel we use the index $\bar{f}, \bar{g}, \bar{h}$ and $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ to denote $(f_1, f_2), (g_1, g_2), (h_1, h_2)$ and $(\alpha_u, \alpha_u), (\beta_u, \beta_u), (\gamma_u, \gamma_u)$, respectively. Also, considering Eq. (12) (Eq. (13)), the spin index of $\psi (\bar{\psi})$ is always a lower one, i.e. $\gamma = 3, 4$ in contrast to $\psi (\bar{\psi})$ which is always upper, i.e. $\gamma = 1, 2$. The normalization factor $1/\sqrt{3}$ is such that at coincident points, $f_1 = f_2$ and for $\kappa = 0$ the two-point function of Eq. (16) below is the identity $(4 \times 4)$ identity matrix $I_4$. We refer to the quark-anti-quark fields in Eq. (12) as the fundamental excitation fields in the individual basis, since each isospin and spin index of the fermion fields appears individually.

The next term in the expansion of Eq. (9) is $\mathcal{G}_{ML}^{(2,1)}$, but because of our parameter restriction, i.e. $\beta_p << \kappa_p$ this is a subdominant term. This term is associated to a vertical plaquette with two bonds, coming from the quark field dependent part of the action, superposed the two vertical sides of the plaquette and in opposite orientation. We note that $\kappa_p^2 \beta_p << \kappa_p^2$ from which we get the restriction $\beta_p / \kappa_p << 1$.

Remark 1 Note that the fields in Eqs. (12) and (13) are local composite fermion fields and gauge invariant (colorless).
We now obtain a fundamental property, called the product structure, which plays a major role in our analysis. In what follows, for simplicity, we drop the superscript notation \((m,0)\) and write simply \((m)\) meaning the coefficient of \(\kappa_p^m\) at \(\kappa_p = \beta_p = 0\). For closure, meaning that correlations on the LHS and RHS of Eq. (10) are the same, we take the fields \(M = M_{\hat{\alpha} \hat{f}}\) and \(L = M_{\tilde{\beta} \tilde{h}}\) in Eq. (10) to obtain, for \(x^0 \leq p < y^0\),

\[
\langle M_{\hat{\alpha} \hat{f}}(u)M_{\tilde{\beta} \tilde{h}}(v) \rangle_T^{(2)} = \sum_{\omega} \langle M_{\hat{\alpha} \hat{f}}(u)M_{\tilde{\beta} \tilde{h}}(p, \bar{w}) \rangle_T^{(0)} \langle M_{\tilde{\beta} \tilde{h}}(p + 1, \bar{w})M_{\tilde{\beta} \tilde{h}}(v) \rangle_T^{(0)},
\]

(14)
i.e. the aforesaid product structure. Similarly to Eq. (14) we have for \(y^0 \leq p < x^0\), taking \(M = M_{\hat{\alpha} \hat{f}}\) and \(L = M_{\tilde{\beta} \tilde{h}}\) in Eq. (11)

\[
\langle M_{\hat{\alpha} \hat{f}}(u)M_{\tilde{\beta} \tilde{h}}(v) \rangle_T^{(2)} = \sum_{\omega} \langle M_{\hat{\alpha} \hat{f}}(u)M_{\tilde{\beta} \tilde{h}}(p + 1, \bar{w}) \rangle_T^{(0)} \langle M_{\tilde{\beta} \tilde{h}}(p, \bar{w})M_{\tilde{\beta} \tilde{h}}(v) \rangle_T^{(0)}.
\]

(15)

Eqs. (14) and (15) can be put together by writing schematically

\[
G_{M\bar{M}}^{(2)}(u, v) = [G_{M\bar{M}}^{(0)} \circ G_{M\bar{M}}^{(0)}](u, v).
\]

This property is our guide to define the two-point function of Eq. (16) and to perform our analysis, i.e. to obtain the mesonic eightfold way from a dynamical perspective.

In agreement with the general definition of Eq. (6) the two-point function for the basic excitation fields is defined by, writing \(G_{M\bar{M}}(u, v) \equiv G_{\ell\ell}(u, v) = G_{\ell\ell}(x = u - v),\)

\[
G_{\ell\ell}(x) = \langle M_\ell(u)M_\ell(v) \rangle_T \chi_{\omega < x^0} + \langle M_\ell(u)M_\ell(v) \rangle_T \chi_{\omega > x^0}
\]

(16)

where the subscripts \(\ell = (\hat{\alpha}, \hat{f})\) and \(\ell' = (\tilde{\beta}, \tilde{h})\) are collective indices. Note that, in Eq. (16), the subtraction \(\langle M\ell(u) \rangle = \langle M\ell(u) \rangle = 0\) is zero using parity symmetry (to be defined ahead). This must be understood whenever we refer to the meson two-point function. The dimension of the two-point matrix, regarding (Spin × Isospin) we have to consider is \((2 \times 3)^2 = 36\).

Concerning Eq. (16) we have the spectral representation, for \(x^0 \neq 0\),

\[
G_{\ell\ell}(x) = \int_{-\infty}^{\infty} \int_{T^d} \lambda_0^{(x^0)} e^{-i\bar{\lambda} \cdot \bar{x}} d(\lambda_0) d(\lambda_0, \lambda_0, \lambda_0, \lambda_0, \lambda_0, \lambda_0) \cdot H, \quad (17)
\]

and is an even function of \(\bar{x}\) by parity symmetry (more details will be given in Appendix A).

**Remark 2** We note that, from the spectral representation of Eq. (17), the fields \(\bar{M}\) create particles in contrast to the fields \(M\) which are auxiliary fields entering the definition of the two-point function of Eq. (16).}

Taking the Fourier transform of Eq. (17), i.e. \(\tilde{G}_{\ell\ell}(p) = \sum_{x \in \mathbb{Z}^d} G_{\ell\ell}(x) e^{-ip \cdot x}\), and after separating the equal time contribution, we obtain

\[
\tilde{G}_{\ell\ell}(p) = \tilde{G}_{\ell\ell}(\bar{p}) + (2\pi)^3 \int_{-1}^{1} f(p^0, \lambda^0) d\lambda^0 d\lambda^0 d\lambda^0 \tilde{G}_{\ell\ell}(\lambda^0) \quad (18)
\]

where

\[
d\lambda^0 d\lambda^0 d\lambda^0 \tilde{G}_{\ell\ell}(\lambda^0) = \int_{T^d} \delta(\bar{p} - \bar{x}) d\lambda^0 d\lambda^0 \tilde{G}_{\ell\ell}(\lambda^0, \lambda_0, \lambda_0, \lambda_0, \lambda_0, \lambda_0) \cdot H, \quad (19)
\]

with \(f(x, y) = (e^{ix} - y)^{-1} + (e^{-ix} - y)^{-1}\), and we set \(\tilde{G}(\bar{p}) = \sum_{\bar{x}} e^{-i\bar{p} \cdot \bar{x}} \tilde{G}(x^0 = 0, \bar{x})\).

Points in the E-M spectrum are detected as singularities of \(\tilde{G}_{\ell\ell}(p)\) on \(\text{Im} p^0\), which are given by the \(w(\bar{p})\) solutions of Eq. (21). To determine the singularities we define by a Neumann series the convolution inverse to the two-point function in the individual spin and isospin basis, namely \(\Lambda = \tilde{G}^{-1}\),

\[
\Lambda = (1 + G_d^{-1} G_d^{-1})^{-1} G_d^{-1} = \sum_{i} (-1)^i \left[ G_d^{-1} G_d^{-1} \right]^i G_d^{-1}
\]

(20)
where $G_d$ is given by $G_{d,\ell}\ell'(u, v) = G_{d,\ell}\ell'(u, u)\delta_{\ell\ell'}\delta_{\alpha\beta}$ with $G = G_d + G_n$.

More precisely, the reason behind the introduction of $\Lambda$ is that it decays faster than $G$. Thus, its Fourier transform $\tilde{\Lambda}(p), \tilde{G}(p)\tilde{\Lambda}(p) = 1$, has a larger analyticity domain in $p^0$ than $\tilde{G}(p)$, which turns out to be the strip $|\text{Im}p^0| < -(4 - \epsilon)\ln\kappa$ (see Ref. [37]). Then,

$$\tilde{\Lambda}^{-1}(p) = |\text{cof} \tilde{\Lambda}(p)|^0/|\det[\tilde{\Lambda}(p)]|,$$

provides a meromorphic extension of $\tilde{G}(p)$. The singularities of $\tilde{\Lambda}^{-1}(p)$ are solutions $\omega(\tilde{p})$ of the equation

$$\det[\Lambda(p^0 = i\omega(\tilde{p}), \tilde{p})] = 0. \quad (21)$$

The solutions $w(\tilde{p})$ will be shown to be the meson dispersion curves and the masses correspond to $w(\tilde{p} = \tilde{0})$.

**Remark 3** The mesons dispersion relations $w(\tilde{p})$ are given by the zeroes of $\det[\Lambda(p^0 = i\omega(\tilde{p}), \tilde{p})] = 0$. Note that, due to the determinant we are free to take any new basis related to the individual spin and isospin basis by an orthogonal transformation.

As we will see in the next section the mesonic eightfold way particles are related to the basic excitation fields of Eq. (12) by a real orthogonal transformation implying that both have identical spectral properties.

To determine the meson masses (dispersion relations) up to and including $O(\kappa^4) (O(\kappa^2))$ we need the global bounds for long distances and the low order in $\kappa$ short distance behavior of $G$ and $\Lambda$. In the sequel, we consider the two-point function of Eq. (16) and its inverse given by Eq. (20) where the spin and isospin indices are in the individual basis. Before we state the Theorems concerning the expansion in $\kappa$ of the two-point function and its convolution inverse we make some remarks. In the $\kappa$ expansion of $G$ we first observe that, each link of the path is composed by an even number of bonds with two by two in opposite orientation. We call intersecting paths those either with links intercepting in one point or superposed links connecting two points on the lattice. For example, in the first case take a path starting at zero and going around a square is an intersecting path at zero; for the second case take $0 \rightarrow x_1 \rightarrow x_1 + x_2 \rightarrow x_1$, with $0, x_1, x_2$ lattice points, has an overlapping link connecting $x_1$ and $x_2$. In our case, for a path with one link we need intersecting paths corresponding to four and six overlapping bonds, with two by two in opposite orientation, connecting the points separated by distance one. For paths with two or more links, as we will determine mass splitting between pseudo-scalar and vector mesons to leading order, which turns out to be $O(\kappa^4)$, our calculation shows that we only need non-intersecting paths. Intersecting paths would give a contribution of $O(\kappa^6)$ or higher to the mass and dispersion curves. For this purpose, we have derived a general formula for calculating non-intersecting paths [see Eq. (B6) in Appendix B]. A direct application of this formula to obtain the low order in $\kappa$ short distance behavior of the two-point function of Eq. (16) shows that the labelling $\ell$ and $\ell'$ in $G_{\ell\ell'}$ is such $\tilde{f} = \tilde{h}$ and is independent of $\tilde{f}$ up to and including $O(\kappa^4)$.

**Remark 4** Using the symmetries of Section III A, we can decompose each nonet into an octet and flavor singlet. The octet states have the same mass, and, up to and including $O(\kappa^4)$ their masses agree with the singlet of isospin. We point out that we can not guarantee, solely based on symmetry considerations, that the equality of masses between octet and singlet states holds to all orders in $\kappa$.

In the sequel we use the convenient notation ($G_{\alpha\beta}$) to denote ($G_{\ell\ell'}$) with $\ell, \ell'$ as in Eq. (16) with fixed $\tilde{f} = \tilde{h}$ and independent of $\tilde{f}$ the $(4 \times 4)$ matrix in the individual spin basis. In a similar way we introduce the matrix ($\Lambda_{\alpha\beta}$). In the sequel we use the ordering $\alpha = 1 = (3, 1), \alpha = 2 = (4, 2), \alpha = 3 = (4, 1)$ and $\alpha = 4 = (3, 2)$. We also let $|\vec{x}| \equiv \sum_{i=1}^3 |x^i|$ and $c$ is an arbitrary constant, which may differ from place to place. Concerning the behavior of $G$ and $\Lambda$ we have the two Theorems given below.

**Theorem 2** $G$ and $\Lambda$ are jointly analytic in $\kappa$ and $\beta$ and satisfy the following global bounds:

1. (22)

$$|G_{\ell\ell'}(x)| \leq c|\kappa|^2|x^0| + 2|\vec{x}|. \quad (22)$$

2. (23)

$$|\Lambda_{\ell\ell'}(x)| \leq \begin{cases} c|\kappa|^2|\kappa| |4(x^0 - 1) + 2|\vec{x}|, & |x^0| \geq 1, \\ c|\kappa|^3|\vec{x}|, & |x^0| = 0. \end{cases} \quad (23)$$
where \( c \) is independent of \( \ell \) and \( \ell' \).

\[ \Box \]

**Proof:** The proof of items 1 and 2 are direct applications of the hyperplane decoupling method. For more details we refer the reader to Ref. [11].

The short-distance behavior of \( \mathcal{G} \) and \( \Lambda \) needed in the proof of Theorem 1 is summarized in the next Theorem.

**Theorem 3** Let \( \rho, \sigma = 0, 1, 2, 3 \), \( \epsilon^0 \) the unitary vector in the temporal direction, \( \epsilon^i, \epsilon^j \) \( (i, j = 1, 2, 3) \), the unitary vectors in the spatial directions and \( \epsilon, \epsilon' = \pm 1 \). The following properties, independent of isospin labelling, hold for \( (\mathcal{G}_{\alpha\beta}) \) and \( (\Lambda_{\alpha\beta}) \), at \( \beta = 0 \):

1. 

\[
\mathcal{G}_{\alpha\beta}(x) = \left\{ \begin{array}{l}
\delta_{\alpha\beta} + \delta_{\alpha\beta} \mathcal{O}(k^8) \\
\delta_{\alpha\beta}k^2 + 2c_2c_6k^6 + \mathcal{O}(k^8) \\
c_2\delta_{\alpha\beta}k^2 + 2c_6k^6 + \mathcal{O}(k^8) \\
\delta_{\alpha\beta}k^4 + \mathcal{O}(k^8) \\
c_2\delta_{\alpha\beta}k^4 + \mathcal{O}(k^8) \\
2c_2k^4 + \mathcal{O}(k^8) \\
c_2\delta_{\alpha\beta}k^4 + \mathcal{O}(k^8) \\
\left[ 2c_2\delta_{\alpha\beta} + (\delta_{\tilde{\alpha}\tilde{\beta}} - \delta_{\tilde{\alpha}}\tilde{\gamma}_c)(1 - \delta_{\tilde{\alpha}\beta}) \right] k^4 + \mathcal{O}(k^8) \\
\left[ 2c_2\delta_{\alpha\beta} + (\delta_{\tilde{\alpha}}\tilde{\gamma}_c + \delta_{\tilde{\gamma}_c} - \delta_{\tilde{\alpha}\beta})k^4 + \mathcal{O}(k^8) \\
c_2\delta_{\alpha\beta}k^4 + \mathcal{O}(k^8) \\
(2c_2\delta_{\alpha\beta} + 2c_2)\delta_{\alpha\beta}k^6 + \mathcal{O}(k^8) \\
\left[ 6c_2\delta_{\alpha\beta} + c_2(\delta_{\tilde{\alpha}}\tilde{\gamma}_c - \delta_{\tilde{\alpha}}\tilde{\gamma}_c - \delta_{\tilde{\alpha}\beta})k^6 + \mathcal{O}(k^8) \\
\left[ 6c_2\delta_{\alpha\beta} + c_2(\delta_{\tilde{\alpha}}\tilde{\gamma}_c + \delta_{\tilde{\gamma}_c} - \delta_{\tilde{\alpha}\beta})k^6 + \mathcal{O}(k^8) \\
\delta_{\alpha\beta}k^6 + \mathcal{O}(k^8)
\end{array} \right. \\
, x = 0; \\
, x = \epsilon^0; \\
, x = \epsilon^1; \\
, x = \epsilon^2; \\
, x = \epsilon^3; \\
, x = \epsilon^0 + \epsilon^1 \epsilon^2; \\
, x = \epsilon^1 + \epsilon^2 \epsilon^3; \\
, x = \epsilon^0 + \epsilon^1 + \epsilon^2 \epsilon^3; \\
, x = \epsilon^0 + \epsilon^1 \epsilon^2 + \epsilon^0 \epsilon^3; \\
, x = 3\epsilon^0.
\]

2. 

\[
\Lambda_{\alpha\beta}(x) = \left\{ \begin{array}{l}
\delta_{\alpha\beta} + (2 + 6c_3)\delta_{\alpha\beta}k^4 + \mathcal{O}(k^8) \\
-\delta_{\alpha\beta}k^2 - \delta_{\alpha\beta}k^6 + \mathcal{O}(k^8) \\
-c_2\delta_{\alpha\beta}k^2 + \mathcal{O}(k^8) \\
\mathcal{O}(k^{10}) \\
(-c_2 + c_2^2)\delta_{\alpha\beta}k^4 + \mathcal{O}(k^8) \\
\mathcal{O}(k^8) \\
(-c_2\delta_{\tilde{\alpha}}\tilde{\gamma}_c + 2c_2^2\delta_{\alpha\beta})k^4 + \mathcal{O}(k^8) \\
(\delta_{\tilde{\alpha}}\tilde{\gamma}_c - \delta_{\tilde{\gamma}_c})(1 - \delta_{\tilde{\alpha}\beta})k^4 + \mathcal{O}(k^8) \\
(\delta_{\tilde{\alpha}}\tilde{\gamma}_c + \delta_{\tilde{\gamma}_c})(1 - \delta_{\tilde{\alpha}\beta})k^4 + \mathcal{O}(k^8) \\
\mathcal{O}(k^8) \\
\mathcal{O}(k^{10}) \\
\mathcal{O}(k^{12}) \\
, x = 0; \\
, x = \epsilon^0; \\
, x = \epsilon^1 + \epsilon^2 \epsilon^3; \\
, x = \epsilon^1 + \epsilon^2 \epsilon^3; \\
, x = \epsilon^1 + \epsilon^2 \epsilon^3; \\
, x = \epsilon^0 + \epsilon^1 \epsilon^2; \\
, x = \epsilon^0 + \epsilon^1 \epsilon^2; \\
, x = \epsilon^0 + \epsilon^1 \epsilon^2; \\
, x = 3\epsilon^0.
\]

\[ \Box \]

**Proof:** The proof is given in Appendix B. The second item of Theorem 3 uses the Neumann series of Eq. (20) using the non-intersecting paths of Eq. (B6).
Remark 5 The action of charge conjugation $C$ (see Appendix A) leave the subspace $\mathcal{H}_M$, i.e. the subspace of $\mathcal{H}_e$ generated by the fields of Eq. (12), stable in the sense that particles and anti-particles states are l.d. and hence, have the same spectral representation. More explicitly, $CM_{31,f} = M_{32,f}$, $CM_{42,f} = M_{41,f}$, $CM_{41,f} = M_{42,f}$, and so considering $\det(p^{\dagger} = iw[p]$; $p) = 0$ they have the same dispersion curves.

Remark 6 Considering Theorem 3 above we note that for $x = 2e^0$, $x = e^0 + e' \epsilon_0$, $x = e^0 + 2e' \epsilon_0$, $x = 2e^0 + e' \epsilon_0$, $x = e^0 + e' \epsilon_0 + e'' \epsilon_0$, $x = 3e^0$ we would expect the contributions $O(\kappa_6)$, $O(\kappa_4)$, $O(\kappa_3)$, $O(\kappa_5)$, $O(\kappa_5)$, $O(\kappa_1)$ respectively, taking into account the global bound $|\Lambda(x)| \leq \varepsilon |x^2 + i|^{2+4|x^2|^{-1}+2|x^2|}$ from Theorem 2. The absence of lower order terms in $\Lambda$ for $|x^0| = 1, 2, 3$, is related to explicit cancellations in the Neumann series and improve the global bounds obtained by the decoupling of hyperplane method.

In the next section, starting from the basic excitation fields of Eq. (12), we introduce a new basis, the total basis, related to the individual spin and isospin basis by a real orthogonal transformation and we show how to make the conventional identification with the pseudo-scalar and vector mesons. We also determine the meson masses, dispersion curves and their multiplicities.

III. PARTICLE BASIS: THE PSEUDO-SCALAR AND VECTOR MESONS FIELDS

This section is divided into four subsections. In the first subsection, we introduce the symmetries at the level of correlations associated to $SU(3)_f$ such as, total isospin, $z$-component of total isospin and total hypercharge. The fact that these symmetries can be implemented as unitary operators in $\mathcal{H}$ is devoted to the next section. The use of orthogonality relations at the level of correlations enable us to show that the two-point function reduce to a block diagonal form. In the second subsection, we introduce spin operators and other symmetries, namely $G$-parity ($G_p$) and spin flip ($F_s$), used to reduce the two-point function to a simpler diagonal form. In the third subsection, we establish the conventional connection with the eightfold way meson states. The final subsection is devoted to the determination of the mesons masses, dispersion curves and their multiplicities.

A. Flavor Symmetry Considerations

Here we define as linear operators on the Grassmann algebra the total isospin, total hypercharge and other operators associated with $SU(3)_f$ symmetry. For fixed $\overline{\alpha}$, $\overline{\mathcal{M}_{\overline{\alpha}f}}$ form a basis for the $3 \otimes 3$ representations of $SU(3)_f$. We define, with $F$ a function of Grassmann fields, suppressing the gauge field dependence,

$$W(U)F = F(\{U\psi\}, \{U\psi\}) \text{, } U\psi = \psi U^\dagger. \quad (26)$$

In Section IV we show how to implement this operator as an unitary operator in the physical Hilbert space $\mathcal{H}$. Letting $U = e^{i\theta F}$ we define the operators $A_j$, $j = 1, 2, \ldots 8$, by

$$A_j F = \lim_{\delta \to 0} \frac{(W(U)-1)}{i\theta} F, \quad (27)$$

where $F_j = \lambda_j/2$ are the traceless self-adjoint Gell-Mann matrices given below

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. $$

For $F = \mathcal{M}_{\overline{\alpha}f}$, $A_j = I_j$, $j = 1, 2, 3$ in Eq. (27), we define components of Isospin (with a $\overline{\alpha}$ meaning complex conjugation)

$$I_j = i_j \times 1 - 1 \times i_j. \quad (28)$$
where $i_j = F_j$ ($j = 1, 2, 3$). Similarly, we have for total hypercharge, $U = e^{iU_F}$,

$$YF = \frac{2}{\sqrt{3}} \lim_{\theta \to 0} \frac{(W(U) - 1)}{i\theta} F_s,$$

from which we get

$$Y = y \times 1 - 1 \times \bar{y}$$

with $y = \frac{2}{\sqrt{3}} F_s$. Also we define the Isospin raising and lowering operators, letting $i_\pm = i_1 \pm i_2$:

$$I_\pm = i_\pm \times 1 - 1 \times i_\mp
= i_\pm \times 1 - 1 \times i_\mp;$$

Total isospin squared:

$$\bar{T}^2 = \sum_{j=1}^3 I_j^2 = \frac{1}{2}(I_+ I_- + I_- I_+ + I_3^2);$$

Raising and lowering operators for SU(3)$_f$:

$$U_\pm = u_\pm \times 1 - 1 \times u_\mp, \quad u_\pm = F_6 \pm iF_8;$$

$$V_\pm = v_\pm \times 1 - 1 \times v_\mp, \quad v_\pm = F_4 \pm iF_3;$$

Quadratic Casimir,

$$C_2 = \sum_{j=1}^{8} A_j^2.$$  

The operators above obey the commutation relations listed below:

$$[I_+, I_-] = 2I_3, \quad [I_3, I_\pm] = \pm I_\pm, \quad [I_j, \bar{T}^2] = 0;$$

$$[Y, I_\pm] = 0, \quad [Y, I_3] = 0;$$

$$[I_3, U_\pm] = \frac{Y}{\sqrt{3}}, \quad [Y, U_\pm] = \pm U_\pm;$$

$$[I_3, V_\pm] = \frac{Y}{\sqrt{3}}, \quad [Y, V_\pm] = \pm V_\pm;$$

$$[C_2, A_j] = 0.$$  

Here we follow the convention of Ref. [3].

From the commutation relations above we can see that $I_\pm$ changes $I_3$ by $\pm 1$, but does not change $Y$; $U_\pm$ changes $I_3$ by $\pm \frac{1}{2}$ and changes $Y$ by $\pm 1$; $V_\pm$ changes $I_3$ by $\pm \frac{1}{2}$, and changes $Y$ by $\pm 1$.

In section IV we show how to lift $\bar{T}^2$, $I_3$, $Y$ and $C_2$ to operators in $\mathcal{H}$ (we recall the notation $\bar{A}$ for an operator in $\mathcal{H}$ if $A$ is the Grassmann algebra operator). We use the eigenvectors of the commuting set $\{\bar{T}^2, I_3, Y, C_2\}$ to form a new basis for each fixed $\bar{a}$. The vectors are denote by $\mathcal{M}_{\bar{a}a}$

$$\mathcal{M}_{\bar{a}a} = \{I, I_3, Y, C_2\}$$

where $\mathcal{L} = \{I, I_3, Y, C_2\}$ denotes the eigenvalues of $\bar{T}^2$, $I_3$, $Y$ and $C_2$; really $I(I + 1)$ is the eigenvalue of $\bar{T}^2$, but we drop this from the notation and for simplicity we write $I$. Explicitly, following the usual procedure to pass from one vector to another inside the nonet we can start, for example with $\psi_{\bar{a},a_{1,1}} \psi_{\bar{a},a_{2,2}}$, and apply the operators $I_\pm$, $U_\pm$, $V_\pm$ to generate 8 basis vectors. These 8 vectors all have the same value of $C_2$ which is conveniently calculated on the vector $U_+ \psi_{\bar{a},a_{1,1}} \psi_{\bar{a},a_{2,2}}$. For this vector $C_2 = \bar{T}^2 + V_3 V_\pm + U_3 U_\pm + U_3 + \frac{3}{2} Y^2$ reduces to $C_2 = \bar{T}^2 + \frac{3}{2} Y + \frac{3}{4} Y^2$ since $V_4$ and $U_4$ are zero acting on this vector and $V_3$ and $U_3$ add to give $3Y/2$. Thus $C_2$ takes the value 3.
Furthermore for the vector
\[ \mathcal{M}_\alpha^0 = \frac{1}{\sqrt{3}} (\bar{\psi}_{a,\alpha,z,u} \psi_{a,\alpha,z,u} + \bar{\psi}_{a,\alpha,z,d} \psi_{a,\alpha,z,d} + \bar{\psi}_{a,\alpha,z,s} \psi_{a,\alpha,z,s}) \]

it is seen that \( \tilde{P}^2, U_+, V_+, Y \) acting on it gives zero, so that \( C_2 \) has the eigenvalue 0.

In this way, for fixed \( \bar{\alpha} = (\alpha_z, a_u) \) we decompose the basis into the direct sum of irreducible representations of \( SU(3)_f \): a one-dimensional flavor singlet (denoted by \( \mathcal{M}_3^0 \), with \( C_2 = 0 \)) and 8-dimensional octet (\( \{ \mathcal{M}_3^k \}_{k=1}^8 \), with \( C_2 = 3 \)) and the labelling distinguishes between them. These states with their quantum numbers are displayed in Fig. 1 where for simplicity we have labelled them by \( \mathcal{M}_3^k \) \( (k = 0, 1, \ldots, 8) \). We list the \( \{ \mathcal{M}_3^k \} \):

\[
\begin{align*}
\mathcal{M}_3^0 &= \frac{1}{\sqrt{3}} (\bar{\psi}_{a,\alpha,z,u} \psi_{a,\alpha,z,u} + \bar{\psi}_{a,\alpha,z,d} \psi_{a,\alpha,z,d} + \bar{\psi}_{a,\alpha,z,s} \psi_{a,\alpha,z,s}) \\
\mathcal{M}_3^1 &= \frac{1}{\sqrt{2}} (\bar{\psi}_{a,\alpha,z,u} \psi_{a,\alpha,z,u} + \bar{\psi}_{a,\alpha,z,d} \psi_{a,\alpha,z,d} - 2 \bar{\psi}_{a,\alpha,z,s} \psi_{a,\alpha,z,s}) \\
\mathcal{M}_3^2 &= \frac{1}{\sqrt{6}} (\bar{\psi}_{a,\alpha,z,u} \psi_{a,\alpha,z,u} - \bar{\psi}_{a,\alpha,z,d} \psi_{a,\alpha,z,d}) \\
\mathcal{M}_3^3 &= \frac{1}{\sqrt{3}} \bar{\psi}_{a,\alpha,z,u} \psi_{a,\alpha,z,d} \\
\mathcal{M}_3^4 &= \frac{1}{\sqrt{3}} \bar{\psi}_{a,\alpha,z,u} \psi_{a,\alpha,z,s} \\
\mathcal{M}_3^5 &= \frac{1}{\sqrt{3}} \bar{\psi}_{a,\alpha,z,d} \psi_{a,\alpha,z,u} \\
\mathcal{M}_3^6 &= \frac{1}{\sqrt{3}} \bar{\psi}_{a,\alpha,z,d} \psi_{a,\alpha,z,s} \\
\mathcal{M}_3^7 &= \frac{1}{\sqrt{3}} \bar{\psi}_{a,\alpha,z,s} \psi_{a,\alpha,z,u} \\
\mathcal{M}_3^8 &= \frac{1}{\sqrt{3}} \bar{\psi}_{a,\alpha,z,s} \psi_{a,\alpha,z,d}.
\end{align*}
\]

The set \( \{ \mathcal{M}_3^k \} \) is related to \( \{ \mathcal{M}_{\bar{\alpha}f} \} \) by a real orthogonal transformation \( B \). Explicitly, using the \( \mathcal{M}_{\bar{\alpha}f} \) ordering \( f = (u, u), (d, d), (s, s), (u, d), (u, s), (d, s), (d, u), (s, d) \) with fixed \( \bar{\alpha} \), \( B \) is giving by

\[ B = B_3 \otimes I_6 \]  

\[
B_3 = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

From the above, recalling the normalization \( \langle \mathcal{M}_{\bar{\alpha}f} \mathcal{M}_{\bar{\alpha}f} \rangle^{(0)} = 1 \) we have \( \langle \mathcal{M}_3^k \mathcal{M}_3^k \rangle^{(0)} = 1 \).

\[
\begin{array}{cccccccc}
 & \mathcal{M}_3^0 & \mathcal{M}_3^1 & \mathcal{M}_3^2 & \mathcal{M}_3^3 & \mathcal{M}_3^4 & \mathcal{M}_3^5 & \mathcal{M}_3^6 & \mathcal{M}_3^7 & \mathcal{M}_3^8 \\
1 & & & & & \mathcal{M}_3^0 & \mathcal{M}_3^1 & \mathcal{M}_3^2 & \mathcal{M}_3^3 & \mathcal{M}_3^4 \\
0 & & & & \mathcal{M}_3^0 & \mathcal{M}_3^1 & \mathcal{M}_3^2 & \mathcal{M}_3^3 & \mathcal{M}_3^4 & \mathcal{M}_3^5 \\
-1 & & & \mathcal{M}_3^0 & \mathcal{M}_3^1 & \mathcal{M}_3^2 & \mathcal{M}_3^3 & \mathcal{M}_3^4 & \mathcal{M}_3^5 & \mathcal{M}_3^6 \\
I_3 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & & & & \\
\end{array}
\]

**Figure 1:** Graphical representation of the tensor product decomposition \( 3 \otimes 3 = 8 \oplus 1 \). \( \mathcal{M}_3^0 \) is the singlet state and the remaining fields \( \{ \mathcal{M}_3^k \}_{k=1}^8 \) are members of the octet.
The two-point function is denoted by, where we recall the convenient notation \( \mathcal{L} = (I, I_3, Y, C_2) \) labelling the quantum numbers of each member of the nonet,

\[
\mathcal{G}_{\beta \bar{\alpha}; \beta \bar{\alpha}}(u,v) = \left\langle \mathcal{M}_{\beta \bar{\alpha}}(u) \mathcal{M}_{\beta \bar{\alpha}}^*(v) \right\rangle \chi_{\alpha \alpha \leq \phi} + \left\langle \mathcal{M}_{\beta \bar{\alpha}}(u) \mathcal{M}_{\beta \bar{\alpha}}(v) \right\rangle^* \chi_{\alpha \alpha > \phi} \tag{39}
\]

which we see decomposes into 8 identical \((4 \times 4)\) blocks and one \((4 \times 4)\) block associated with the octet and flavor singlet, respectively. We remark that for the two-point function of Eq. (39) the product structure still holds, because the states \(\{\mathcal{M}_{\beta \bar{\alpha}}\} \) are related to \(\{\mathcal{M}_{\beta \bar{\alpha}}\} \) by an orthogonal transformation [see Eqs. (37) and (38)], and we still have the faster decay in \(\kappa\) for the convolution inverse (see Theorem 2).

\section*{B. Block Diagonalization of the Two-point Function: Spin Operators and Other Symmetries}

Now we want to further reduce these \((4 \times 4)\) blocks of \(\mathcal{G}\) using additional symmetries. Then we will make conventional identification with meson particles. Here we list the symmetries and leave details for Appendix A. The time reversal symmetry, namely, permutations of flavor indices. For \(F\) a function of the field algebra we define the linear operator \(\mathcal{G}_p\) by, suppressing the gauge fields,

\[
\mathcal{G}_p F_c (\bar{\psi}, \psi) = F_c (P \bar{\psi}, \psi P) \tag{40}
\]

where \(F_c (\bar{\psi}, \psi) = CF(\bar{\psi}, \psi), C\) is the charge conjugation linear operator and \(P \in SU(3)_f\) is one of the six permutations of the flavor indices, i.e. of \(u, d, s\). For example, if \(P\) permutes \(u\) and \(d\) we take

\[
P = P_{ud} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{41}
\]

\(\mathcal{G}_p\) is implemented by an unitary operator on \(\mathcal{H}\) (see Section IV). We use this symmetry to further decompose the space of excitations and which will be seen to correspond to the decomposition into pseudo-scalar and vector excitations. Taking \(P\) to be the permutation of \((f_1, f_2)\). \(\mathcal{G}_p\) acts on \(\mathcal{M}_{\beta \bar{\alpha}}\) as

\[
\begin{align*}
\mathcal{M}_{31, f_1 f_2} &\rightarrow \mathcal{M}_{42, f_1 f_2} \\
\mathcal{M}_{42, f_1 f_2} &\rightarrow \mathcal{M}_{31, f_1 f_2} \\
\mathcal{M}_{41, f_1 f_2} &\rightarrow \mathcal{M}_{32, f_1 f_2} \\
\mathcal{M}_{32, f_1 f_2} &\rightarrow \mathcal{M}_{41, f_1 f_2}.
\end{align*} \tag{42}
\]

For fixed \(\mathcal{L}\) we decompose the space \(\mathcal{M}_{\beta \bar{\alpha}}\) into eigenvectors of \(\mathcal{G}_p\) given by (suppressing all but the spin index)

\[
\begin{align*}
\frac{1}{\sqrt{2}} (\mathcal{M}_{31} + \mathcal{M}_{42}), \quad \text{with eigenvalue } 1 \\
\frac{1}{\sqrt{2}} (\mathcal{M}_{31} - \mathcal{M}_{42}), \quad \mathcal{M}_{41}, \quad \text{with eigenvalue } -1. 
\end{align*} \tag{43}
\]

We denote the vectors in Eq. (43) by \(\mathcal{M}_{\beta \bar{\alpha}}\) which are related to \(\mathcal{M}_{\beta \bar{\alpha}}\) by a real orthogonal transformation explicitly given by

\[
A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \tag{44}
\]

recalling the ordering for \(\alpha\) such as \(3, 1\), \(4, 2\), \(4, 1\) and \(3, 2\).

We now turn to the definition of the spin operator. In the continuum we identify the components of the total angular momentum with the generators of infinitesimal rotations about coordinate axis. Starting from a state \(\Phi(x)\) created by local single or composite field we consider a zero spatial momentum improper state \(\Phi_0 = \int \Phi(x) d\vec{x}\). \(\Phi_0\)
is expected to have zero spatial angular momentum, i.e. only spin angular momentum, and the rotation operator reduces to a rotation in spin space.

On the lattice a $\pi/2$ rotation about any one of the coordinate axis $x, y, z$ is a symmetry giving rise to a unitary operator

$$\hat{Y}(U) = \int_{-\pi}^{\pi} e^{i\lambda} dE(\lambda)$$

which is a lift from the linear transformation

$$\mathcal{Y}(U) = F(\{U\psi\}(x_r), \{\bar{U}\psi\}(x_r))$$

with $x_r$ denoting the coordinates of the rotated point and $U = U_2 \oplus U_2 \equiv e^{i\theta j}$ where $U_2 = e^{i\frac{\pi}{2} \sigma_{x,y,z}}$, $\theta = \pi/2$. By the spectral theorem we can define $\hat{M}_{x,y,z}$, the components of a lattice total angular momentum, by

$$\hat{M} = \frac{2}{i\pi} \ln \hat{Y}(U) = \frac{2}{i\pi} \int_{-\pi}^{\pi} \lambda dE(\lambda).$$

If we consider the zero spatial momentum improper state $\Phi_0 = \sum_x \Phi(x)$ (expected to have only spin angular momentum) for the special case, omitting all indices,

$$\mathcal{Y}(U)\Phi_0 = \sum_x (U\psi)(\bar{U}\psi)(\vec{x})$$

such as only the spin space is transformed and the total angular momentum is expected to reduce to spin angular momentum only. We define $J_{x,y,z}$, the component of total spin, acting on $\Phi_0$ by

$$J\Phi_0 = \frac{2}{i\pi} \ln \hat{Y}(U)\Phi_0 = \sum_x [(j\psi)\psi - \bar{\psi}(j\bar{\psi})](x)$$

where we have used the spectral theorem with $U = e^{i(\pi/2)j} = \sum \psi P_\lambda$ and for a function $f(w) = \sum a_w w^n$, 

$$f(\mathcal{Y}(U))\Phi_0 = \sum_x \sum_{\lambda_1, \lambda_2} f \left( e^{i(\pi/2)(\lambda_1 - \lambda_2)} \right) P_{\lambda_1} \bar{\psi} P_{\lambda_2} \psi.$$

The argument of $\ln$ is well-defined for $(\max |J_z|)\pi/2 < \pi$ which includes the meson states, i.e. $|J_z| = 0, 1$. On the other hand $J\Phi_0$ is precisely what we obtain from the continuous rotation limit

$$\lim_{\theta \to 0} \frac{(Z(U) - 1)}{i\theta}\Phi_0$$

where $Z(U) = F(\{U\psi\}, \{\bar{U}\psi\})$.

$J_{x,y,z}$ obey the usual angular momentum algebra. Below we will show that the correlations for different spin states of zero momentum states, within a member of the nonet, are related by the usual raising and lowering operations which implies that the masses of the $(J, J_z) = (1, 1)$ and $(J, J_z) = (1, 0)$ spin states are equal.

The vectors in Eq. (43) are eigenvectors of $\hat{J}^2$ and $J_z$ with eigenvalues $(0, 0), (1, 1), (1, 0), (1, -1)$, respectively. It turns that these eigenvalues and eigenvectors in the total spin basis correspond to those of $G_\rho$. It should be noted that $G_\rho$ does not distinguish among the states $(J, J_z) = (1, 1), (1, 0), (1, -1)$. We will label the vectors in Eq. (43) by $\mathcal{M}_J$ with this total spin notation $J = (J, J_z)$ in the order given above. The associated two-point function in the total spin basis is denoted by $G_{J,J'}(x)$ and has the structure

$$G_{J,J'}(x) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & d \\ 0 & c^* & e & f \\ 0 & d^* & f^* & g \end{pmatrix}$$

with $a, b, c, g$ real.
We obtain further relations among the elements of $G_{\mathcal{J}\mathcal{T}'}(x)$ by exploiting a new local anti-linear symmetry which we call spin flip, denoted by $\mathcal{F}_s$. This symmetry is the composition

$$\mathcal{F}_s = -i\mathcal{T}\mathcal{C}\mathcal{T}$$

(48)

where $\mathcal{T}$ is time reversal, $\mathcal{C}$ charge conjugation and $\mathcal{T}$ time reflection (see Refs. [20, 21, 30]) symmetries. $\mathcal{F}_s$ can be implemented as an anti-unitary transformation in $\mathcal{H}$ (see Refs. [20, 21]). These symmetries are discussed in more detail in Appendix A. Using this symmetry we obtain the structure

$$G_{\mathcal{J}\mathcal{T}'}(x) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c^* & e \\ 0 \end{pmatrix}.$$  

(49)

It is interesting to observe the symmetry of $G_{\mathcal{J}\mathcal{T}'}$ about the secondary diagonal in the lower right $(3 \times 3)$ block. The convolution inverse, denoted by $\Gamma_{\mathcal{J}\mathcal{T}'}(x)$, has the same structure of Eq. (49). There is a partial restoration of continuous rotational symmetry. In particular, fixing a nonet member, we show that the zero momentum 2-point correlation diagonal elements are identical for fixed $J$ and all $J_z$. Of course, in the continuum, for model with rotational symmetry, fixing the energy, this is true for all spatial momentum. We already know from previous symmetry considerations, i.e. the spin flip symmetry, that they only depend on $|J_z|$.

To see the lattice result write $\tilde{\psi}\tilde{\psi}$ for a typical term of a meson creating field. By the $\pi/2$ spatial rotation symmetry we have, with $u = (u_0, \vec{0})$,

$$\langle \psi\bar{\psi}(u)\bar{U}\bar{\psi}(v)\rangle = \langle (\psi\bar{U})(\psi\bar{U})(\psi\bar{\psi}(v)) \rangle$$

where $v_r = (v^0, \vec{v})$ is the rotated point. Summing over $\vec{v}$, $\psi\bar{v}$ can be replaced by $\tilde{v}$ and the identity holds for powers of $U$ and for linear combinations of correlations. Using the spectral theorem for $U = \sum \lambda e^{i\pi/2}P_\lambda$ as before and for a function $f(z) = \sum a_n z^n$ we obtain

$$\sum \langle \tilde{\psi}\tilde{v}(u) \sum f(e^{i\pi/2}\lambda_1 - \lambda_2)(\bar{U}\bar{\psi})(U\bar{\psi}) \rangle = \sum \langle \psi\bar{U}U\bar{\psi}(u)f(e^{i\pi/2}(\lambda_1 - \lambda_2))\psi\bar{\psi}(v) \rangle.$$  

For the function $f(x) = 2/(i\pi) \ln x$ we get

$$\sum \langle \psi\bar{v}(u)J\tilde{\psi}\tilde{v}(v) \rangle = \sum \langle \psi\bar{\psi}J_{12}(u)\tilde{\psi}\tilde{\psi}(v) \rangle$$

where $J = j \times 1 - 1 \times j$, $j = j_x j_y j_z$.

Multiply by $\tilde{w}_{34}$, $v_{12}$ and sum over components to write

$$(Jw, Rw) = (w, RJv)$$

where we have used the usual complex Hilbert space notation and that $J$ is self-adjoint. In the above $(Jw, Rw) = (Jw)_{\alpha}R_{\beta\alpha}^{\nu\beta}$ where $R_{\beta\alpha} = \sum_{\alpha'}(M_{\beta\alpha}(u)M_{\alpha'\beta}(v))$. Thus, letting $J_{+} = J_x + iJ_y$, we have

$$(J_+ w, Rw) = (w, RJ_+v)$$

and taking $v = \chi_{J_+, J_+} - 1$, $w = \chi_{J_+ J_+} - 1$ where $\chi_{Jm}$ is the normalized eigenfunction of total spin $J$ and $z$-component $m$ we get

$$(\chi_{J_+, J_+} - 1, R_{\chi_{J_+, J_+}} - 1) = (\chi_{J_+, J_+}, R_{\chi_{J_+, J_+}})$$

so that for diagonal elements of the nonets $\sum_{\bar{x}} G_{J_+, J_+}(x^0, \bar{x}) = \sum_{\bar{x}} G_{J_+, J_+} - 1(x^0, \bar{x})$,

$$\tilde{G}_{J_+, J_+}(p^0 = \vec{p} = 0) = \tilde{G}_{J_+, J_+} - 1(p^0 = \vec{p} = 0)$$

and the same for $\tilde{G}_{J_+, J_+} - 1(p^0 = \vec{p} = 0)$.

For $\vec{p} \neq 0$, the determinant of $\tilde{G}_{J_+}(p^0 = i\chi, \vec{p})$ factorizes using the formula for the roots of a cubic. For $\vec{p} = \vec{0}$, $\tilde{G}_{J_+}(p^0 = i\chi, \vec{p} = 0)$ is diagonal using additional symmetries of $\pi/2$ rotations about $e_3$. We can obtain further relations among the diagonal elements in $\tilde{G}_{J_+}(p^0 = i\chi, \vec{p} = 0)$, i.e. for $\tilde{G}_{J_+} = \tilde{G}_{J_+}, \tilde{G}_{(1,1)} = \tilde{G}_{(1,-1)}$, using the symmetry of reflection in $e_1$ (for more details see Appendix A). It is interesting to observe that, to obtain these properties for $\vec{p} = \vec{0}$, we do not need spin flip symmetry $\mathcal{F}_s$. We use the auxiliary function method in subsection III D to obtain convergent expansions for the masses $M = M(\kappa)$ in $\det \tilde{G}_{J_+}(p^0 = iM, \vec{p} = \vec{0}) = 0$. 


C. Particle Identification and Basic Excitation States

We make some remarks about the basis for the octet and flavor singlet and the identification of particles. Applying \( I_- (I_+ \) to \( \bar{\psi}_u \psi_d \) (\( \bar{\psi}_d \psi_u \)), identified with the \( \mathcal{M}^3 \) (\( \mathcal{M}^0 \)) state of Fig. 1, where we suppress spin, gauge indices for simplicity, we obtain \( \bar{\psi}_u \psi_u - \bar{\psi}_d \psi_d \); applying \( U_+ \), \( U_- \), \( V_+ \), \( V_- \) to the outer 4 octet members (i.e. to \( \mathcal{M}^8 \), \( \mathcal{M}^0 \), \( \mathcal{M}^2 \), \( \mathcal{M}^4 \) in Fig. 1, respectively) on the outer rim we generate the vectors \( \bar{\psi}_u \psi_u - \bar{\psi}_d \psi_d \) and \( \bar{\psi}_u \psi_u - \bar{\psi}_u \psi_d \). These three vectors are l.d. and we can take the linear combinations

\[
\begin{align*}
 w_1 &= \bar{\psi}_u \psi_u - \bar{\psi}_d \psi_d \\
 w_2 &= 2\bar{\psi}_u \psi_u - \bar{\psi}_u \psi_d - \bar{\psi}_d \psi_d
\end{align*}
\]

identified with \( \mathcal{M}^2 \) and \( \mathcal{M}^4 \), respectively, together with the six outer rim vectors to form a basis for the 8 dimensional representation of \( SU(3)_f \) in the decomposition \( 3 \otimes 3 = 8 \oplus 1 \). The quantum numbers of \( w_1, w_2 \) are

\[
\begin{align*}
 w_1 : I &= 1, I_3 = 0, Y = 0, c_2 = 3, \\
 w_2 : I &= 0, I_3 = 0, Y = 0, c_2 = 3.
\end{align*}
\]

On the other hand the basis vector for the one dimensional representation is the flavor singlet (\( \mathcal{M}^0 \))

\[
w_0 = \bar{\psi}_u \psi_u + \bar{\psi}_d \psi_d + \bar{\psi}_u \psi_u
\]

with quantum numbers

\[
w_0 : I = 0, I_3 = 0, Y = 0, c_2 = 0
\]

so that the three vectors \( w_0, w_1, w_2 \) have distinct quantum numbers.

We now consider the identification of physical particles for broken \( SU(3)_f \). The outer rim vectors are identified with particles as well as \( w_0 = \eta' \), \( w_1 = \pi^0 \) and \( w_2 = \eta \) for pseudo-scalar mesons; for vector mesons \( w_1 \) is identifies with \( \rho^0 \) and the \( \psi \) and \( \omega \) seems to be best described as a strong mixtures of \( w_0 \) and \( w_2 \). In order to make the conventional identification of states presented in Fig. 1 with particles, in Figs. 2 and 3 below are depicted the pseudo-scalar and vector mesons along with the associated quantum numbers. Referring to Fig. 1 we see that \( \mathcal{M}^3 \) \( \to \eta', \mathcal{M}^1 \to \eta, \mathcal{M}^0 \to \pi^0, \mathcal{M}^2 \to \pi^+ \) and so on. Note that since the Baryon number for the mesons is zero we get \( S = Y = B = Y \).

The charges verify the Gell-Mann and Nishijima relation \( Q = I_3 + Y/2 \).

Note that \( \eta' \), \( \eta \) and \( \pi^0 \) are invariant under charge conjugation. Also, \( C\pi^\pm = \pi^\mp \), \( CK^\pm = K^\mp \) and \( CK^0 = K^0 \). Hence, charge conjugation \( C \) changes the sign of the hypercharge \( Y \) and the third component of isospin \( I_3 \) of a pseudo-scalar state.

\[
\begin{array}{cccccc}
 Y & & K^0 & & K^+ & & S & I \\
 1 & & \hline & & \hline & & 1 & 1/2 \\
 0 & \pi^- & \eta & \eta' & \pi^+ & 0 & 1, 0 \\
 -1 & K^- & K^0 & \hline & \hline & -1 & 1/2 \\
 I_3 & -1 & -1/2 & 0 & 1/2 & 1
\end{array}
\]

**Figure 2:** The pseudo-scalar mesons (\( J = 0 \)). The Hypercharge (\( Y \)), Strangeness (\( S \)), Isospin total (\( I \)), z-component of Isospin (\( I_3 \)) are indicated.
w auxiliary matrix function in our determination of meson masses and dispersion curves, due to our restriction with $\mathcal{J}$ and the auxiliary function $\mathcal{W}$ without approximation, we make a nonlinear transformation from $\mathcal{J}$ for this we need the short distance behavior of $\mathcal{G}$ and $\Lambda$ of Theorem 3. Fixing a member of the octet or singlet the mass-determining equation is $\det\tilde{\mathcal{G}} = 0$ and for notational simplicity we write $\tilde{\mathcal{G}}$. We note that the same global bounds of Theorem 2 hold for $\Gamma_{J'}(x)$. The short distance behavior of $\Gamma_{J'}(x)$ is related to the behavior in the individual spin basis given in Theorem 3 by the similarity transformation with the orthogonal transformation $B$ given in Eqs. (37) and (38).

The solution of $\det\tilde{\Gamma}_{J'} = 0$ for all $\vec{p}$ runs out to infinity as $\kappa$ goes to zero. To find the solutions of $\det\tilde{\Gamma}_{J'} = 0$ without approximation, we make a nonlinear transformation from $\vec{p}$ to an auxiliary variable $w$ and introduce an auxiliary matrix function $H_{J,J'}(w,\kappa,\vec{p})$ (for $\vec{p} = 0$ we have the mass) to bring the solution for the nonsingular part $w(\vec{p}) + 2\ln\kappa$ of the dispersion curves from infinity to close to $w = 0$ for small $\kappa$. With this function we can cast the problem of determining dispersion curves and masses into the framework of the analytic implicit function theorem. To this end we introduce the new variable, with $c_2(\vec{p}) \equiv c_2 \sum_{i=1}^{2} \cos p^i$ and we recall from Theorem 3 that $c_2 = 1/4,\]
\[w = 1 - c_2(\vec{p})\kappa^2 - \kappa^2 e^{-ip^0}\]

and the auxiliary function $H_{J}(w,\kappa,\vec{p})$ such that
\[\tilde{\Gamma}_{J}(p^0,\vec{p}) = H_{J} \left( w = 1 - c_2(\vec{p})\kappa^2 - \kappa^2 e^{-ip^0}, \kappa, \vec{p} \right)\]
where $H_{J}(w,\kappa,\vec{p})$ is defined by, using $\Gamma(x^0,\vec{x}) = \Gamma(-x^0,\vec{x})$ (see Lemma 1 Item 2 in Appendix A),
\[H_{J,J'}(w,\kappa,\vec{p}) = \sum_{\vec{x}} \Gamma_{J,J'}(x^0 = 0,\vec{x}) e^{-ip^0\vec{x}} + \sum_{n \geq 1, \vec{x}} \Gamma_{J,J'}(x^0 = n,\vec{x}) e^{-ip^0\vec{x}}\]
\[\times \left[ \frac{1 - w - c_2(\vec{p})\kappa^2}{\kappa^2} \right]^n + \left( \frac{\kappa^2}{1 - w - c_2(\vec{p})\kappa^2} \right)^n\]
with $J = J'$. By the global bounds of Theorem 2, $H$ is jointly analytic in $w$ and $\kappa$ for $|w|$ and $|\kappa|$ small.
The mass determining equation become

\[ H_J(w, \kappa) \equiv H_J(w, \kappa, \bar{p} = \bar{0}) = 0. \]  

(52)

In the sequel we determine the masses up to and including \( O(\kappa^4) \).

**Remark 1** We have not found any symmetry that allow us to show that the flavor singlet masses are the same as the octet masses. Our calculation shows that their masses are the same up to and including \( O(\kappa^4) \).

For convenience we separate the time zero, and one and the remaining contributions to \( H_J = H_J(w, \kappa) \) and we use the short distance behavior of \( \Gamma_J \) in the total spin basis (see Eq. (49)). The contributions are,

\[ n = 0: (1 + c_0 \kappa^4) - c_2(\bar{0}) \kappa^2 + c_4(\bar{0}) c_4 \kappa^4 + a_J \kappa^4 + O(\kappa^6) \]

where \( c_0 = 2 + 6 \epsilon_2^2, c_4 = c_2 - 1 \) and \( a_J \kappa^4 \) are the \( \kappa^4 \) contributions from all points of the form \( x = e^i + e^j \), \( i, j = 1, 2, 3 \) called angle contributions.

\[ n = 1: -\kappa^4 - (1 - w - c_2(\bar{0}) \kappa^2) = \frac{\kappa^4}{(1 - w - c_2(\bar{0}) \kappa^2)} + O(\kappa^6) \]

\[ n \geq 2: O(\kappa^6). \]

Thus we can write \( H_J(w, \kappa) \) in the form, with \( b_J = -1 + c_0 + c_2(\bar{0}) c_4, \)

\[ H_J(w, \kappa) = w + b_J \kappa^4 - \frac{\kappa^4}{1 - w - c_2(\bar{0}) \kappa^2} + a_J \kappa^4 + \kappa^6 r_J(w, \kappa) \]

where \( r_J(w, \kappa) \) is jointly analytic in \( w \) and \( \kappa \). We see that \( H_J(0, 0) = 0 \) and \( (\partial H_J/\partial w)(0, 0) = 1 \) so that the analytic implicit function theorem applies and yields the analytic function \( w_J(\kappa) \) such that

\[ H_J(w_J(\kappa), \kappa) = 0. \]  

(53)

The solution \( w_J(\kappa) \) has the form

\[ w_J(\kappa) = \kappa^4 - (a_J + b_J) \kappa^4 + O(\kappa^6). \]

Returning to Eq. (50) the mass is given by

\[
M_J = \text{ln} e^{-i(p^2 = M_J)} = -2 \ln \kappa + \ln(1 - w_J - c_2(\bar{0}) \kappa^2) = -2 \ln \kappa + (-1 + c_0 + 6c_2c_4) \kappa^4 - 6c_2 \kappa^2 - \frac{1}{2} \epsilon_2^2 \kappa^4 + a_J \kappa^4 + O(\kappa^6). \]

(54)

Suppressing, in what follows, the subscript \( J \) from the notation, the implicitly defined \( w_J(\kappa) \), for each \( J \), has an explicit representation in terms of the Cauchy integral,

\[ w(\kappa) = \frac{1}{2\pi i} \int_{|w|<\alpha} w \frac{\partial H}{\partial w}(w, \kappa) dw, \]

(55)

where \( \alpha > 0 \) is sufficiently small (see Ref. [38]). From this representation we see that \( w(\kappa) \) is analytic in \( \kappa \). We note that the integral formula of Eq. (55) permits us to deduce an explicit formula for \( w_n = (1/n)!d^n w(0)/ds^n \), the \( n \)-th Taylor coefficient of the analytic function \( w(\kappa) \), implicitly defined by Eq. (53). For the general procedure to obtain \( w_n \) from Eq. (55) we refer the reader to Ref. [39].

Noting that \( a_J = 4 \left[ \Gamma_J(e^1 + e^2) + \Gamma_J(e^1 + e^3) + \Gamma_J(e^2 + e^3) \right] \) we find

\[ a_J = \begin{cases} 
(-3 + 4 \epsilon_2^2) \kappa^4 = -3 \kappa^4/2, & J = 0 \\
(-1 + 4 \epsilon_2^2) \kappa^4 = \kappa^4/2, & J = 1, J_z = 0, -1, 1.
\end{cases} \]

(56)

Thus we see there is a mass splitting between the total spin one and total spin zero states given by

\[ M_{(1,J_z)} - M_{(0,0)} = 2 \kappa^4 + O(\kappa^6). \]
For $\beta \neq 0$, $\beta \ll \kappa$, the arguments above hold and the non-singular contribution to the mass is jointly analytic in $\kappa$ and $\beta$. In particular, the mass splitting persists for $\beta \neq 0$. The implicit function $w(\kappa, \beta)$ is given by the above integral representation of Eq. (55) making the obvious replacements $H(w(\kappa))$ by $H(w(\kappa, \beta))$ in the integrant.

We now turn to the determination of dispersion curves. We recall the block decomposition of $G_{\mathcal{J}_3\mathcal{J}_3}$ in Eq. (49) which is the same as $\Gamma_{\mathcal{J}_3\mathcal{J}_3}$. We write $\Gamma_{\mathcal{J}_3\mathcal{J}_3} = D_3 \oplus D_1 (\beta, \kappa, |\mathcal{J}_3\mathcal{J}_3|)$ a $n \times n$ matrix which implies that, for the pseudo-scalar meson, i.e. the number $D_1$, we can still apply the auxiliary function method to determine the dispersion curves $w_\rho(\vec{p})$. They are given by

$$w_\rho(\vec{p}) = -2 \ln \kappa - 6c_2 \kappa^2 + c_2 \kappa^2 \sum_{j=1}^{3} 2(1 - \cos p^j) + \kappa^4 r_\rho(\kappa, \vec{p}),$$

with $r_\rho(\kappa, \vec{p})$ jointly analytic in $\kappa$, $\text{Im}p^j$ for $|\kappa|$, $|\text{Im}p^j|$ small.

Concerning the $(3 \times 3)$ block, for fixed $\kappa$ and $\vec{p}$, we can apply a Rouché’s theorem argument to the analytic function

$$f(w) \equiv \text{det} H(w, \kappa) = \text{det}(wI_3) + [\text{det} H(w, \kappa) - \text{det}(wI_3)]
= g(w) + h(w).$$

In the disc $|w| \leq c \kappa^4$, $c \gg 1$, $g(w) > h(w)$, so that the equation $f(w) = 0$ and $g(w) = 0$ have the same number of solutions. Since $g(w) = w^3 = 0$ has 3 solutions we see that $g(w) = 0$ has exactly 3 not necessarily distinct solutions for the dispersion curves.

From the relation between $w$ and $p^0$ of Eq. (50) the dispersion curves are given by,

$$w(\vec{p}) = -2 \ln \kappa - 6c_2 \kappa^2 + c_2 \kappa^2 \sum_{j=1}^{3} 2(1 - \cos p^j) + \mathcal{O}(\kappa^4).$$

We still do not know if the dispersion curves are the same for the singlet flavor and octet.

Recalling Eq. (49) we note that for the vector mesons we can use the Cardano’s formula for the roots of a cubic to obtain factorization of the $3 \times 3$ determinant. Each factor is a sum of terms with square and cubic roots of polynomials of maximum degree 6 in the matrix elements of $\bar{\Gamma}$. The expressions are lengthy and we won’t present them here but they can be viewed using standard mathematical programs such as Maple. The presence of the square of a cubic roots and the lack of knowledge of the order in $\kappa$ where $\bar{\Gamma}_{00} - \bar{\Gamma}_{11}$ is non-zero prevent us from solving the equations for the dispersion curves using the auxiliary function method.

IV. ISOSPIN, HYPERCHARGE, SPIN FLIP AND $G_\rho$ OPERATORS IN $\mathcal{H}$

In this section, following Refs. [20, 21] and inspired by the treatment of point groups given in Refs. [23], we construct the total isospin, total hypercharge, $G_\rho$ and spin flip, defined in Eqs. (28), (29), (40), (48), respectively, as self-adjoint operators acting on the physical Hilbert space $\mathcal{H}$. The spin flip symmetry and its implementation by an anti-unitary operator is treated in Refs. [20, 21]. The spin operators are defined on the field algebra but not in the physical Hilbert space. For a function $F$ on the field algebra and $U \in SU(3)$ we define a linear operator $W(U)$ by Eq. (26). Using the F-K formula, and for functions $F$ and $G$ of the basic fields, of finite support, we define the Hilbert space operator $W(U)$ by the sesquilinear form

$$(G, W(U)F)_{\mathcal{H}} = \langle [W(U)]F|G \rangle.$$ 

so that, using Eq. (26), we have

$$(G, W(U)F)_{\mathcal{H}} = \langle F(\{\psi U^\dagger\}, \{U\bar{\psi}\})|G(\{\psi\}, \{\bar{\psi}\}) \rangle
= \langle F(\{\psi\}, \{\bar{\psi}\})|G(\{\psi U\}, \{U^\dagger \bar{\psi}\}) \rangle = \langle F\Theta W(U)G \rangle
= \langle W(U^\dagger)G, F \rangle_{\mathcal{H}},$$

so that $W(U)^\dagger = W(U^\dagger)$, and where we have used the $SU(3)_f$ symmetry on the correlation functions in the rhs. Furthermore, we have

$$(W(U)G, W(U)F)_{\mathcal{H}} = \langle [W(U)]F|\Theta[W(U)]G \rangle = \langle F\Theta G | (G, F)_{\mathcal{H}} \rangle.$$
Hence, $\mathcal{W}(U)$ is an isometry, i.e. $\mathcal{W}(U)\mathcal{W}(U) = 1$. Interchanging $U$ and $U^\dagger$, we also have $\mathcal{W}(U)\mathcal{W}(U)^\dagger = 1$. Then, $\mathcal{W}(U)^\dagger$ is also isometric which implies that $\mathcal{W}$ is unitary. The isometry property of $\mathcal{W}(U)$ is seen by first considering $F$ and $G$ monomials and then extending to $F$ and $G$ elements of $\mathcal{H}$ by continuity. That $\mathcal{W}$ in Eq. (59) is well defined follows from the fact that taking $F \in \mathcal{F}$ (recall $\mathcal{F}$ denotes the set of nonzero $F$ such that $\langle F\Theta F \rangle = 0$) then if $F \in \mathcal{F}$, $\mathcal{W}(U)F$ is also in $\mathcal{F}$. We note that $\mathcal{G}_0$ is a composition of unitary operators (charge conjugation and discrete flavor permutations) and can lift to a unitary operator in $\mathcal{H}$. Generators $A_j$ associated with the 8 one parameter subgroups of $SU(3)_f$ are self-adjoint by Stone’s theorem Refs. [36].

We see that $\mathcal{W}(U)$ commutes with time evolution $\mathcal{T}_0$ by noting that $\mathcal{W}(U)\mathcal{T}_0a\mathcal{W}(U)^\dagger = \mathcal{T}_0a$. Thus, the $SU(3)_f$ generators defined as operators in $\mathcal{H}$ also commute with $\mathcal{T}_0$.

We now turn to the spin flip symmetry defined in Refs. [20, 21, 30]. From a consideration of the composition of the symmetries of time reversal $\mathcal{T}$, charge conjugation $\mathcal{C}$ and time reflection $\mathcal{T}$ given in Appendix A, the local spin flip operator is defined by $\mathcal{F}_s = -i\mathcal{T}\mathcal{C}\mathcal{T}$ which acts on single Fermi fields by $\psi_\alpha(x) \rightarrow A_{\alpha\beta}\psi_\beta(x)$, $\bar{\psi}_\alpha(x) \rightarrow \bar{\psi}_\beta(x)B_{\beta\alpha}$ where $A = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$ is real anti-symmetric and $B = A^{-1} = -A$. For functions of the gauge fields, $f(g_{xy}) \rightarrow f(g_{yx})$, where $*$ denotes the complex conjugate. More explicitly, $\tilde{\psi}_1 \rightarrow \tilde{\psi}_2$, $\tilde{\psi}_2 \rightarrow -\tilde{\psi}_2$, $\tilde{\psi}_3 \rightarrow \tilde{\psi}_4$ and $\tilde{\psi}_4 \rightarrow -\tilde{\psi}_3$.

In more detail, if $F$ is a polynomial (not necessarily local) in the gauge and Fermi fields, and suppressing the lattice site arguments,

$$F = \sum a_\ell mng_\ell \bar{\psi}_m \psi_n,$$

then extend $\mathcal{F}_s$ by

$$\mathcal{F}_sF = \sum a_\ell mng_\ell (\psi B)^m (\bar{\psi} A)^n,$$

and take $\mathcal{F}_s$ to be order preserving. Note that $\mathcal{F}_s$ is anti-linear but not in the same sense as $\Theta$ (no complex conjugation of $g$ for $\mathcal{F}_s$).

With these definitions, the action of Eq. (2) is termwise linear and the symmetry operation is a symmetry of the system satisfying $\langle F \rangle = \langle \mathcal{F}_s F \rangle$. For the $(4 \times 4)$ block of $G$, applying $\mathcal{F}_s$ gives the structure presented in Appendix A. For the implementation of $\mathcal{F}_s$ as anti-unitary operator in $\mathcal{H}$ we refer the reader to Refs. [20, 21].

V. UPPER GAP PROPERTY AND EXTENSION OF THE SPECTRAL RESULTS FROM $\mathcal{H}_M$ TO ALL $\mathcal{H}_e$

Up to now, we have considered the spectrum generated by vectors in $\mathcal{H}_M \subset \mathcal{H}_e$. As in Refs. [10, 11], we use a correlation subtraction method (see Ref. [22]) to show that the eightfold way meson spectrum is the only spectrum in all $\mathcal{H}_e$, up to near the two-meson threshold of $\approx -4 \ln \kappa$. For $L \in \mathcal{H}_e$ we have the spectral representation and F-K formula (with $P_{11}$ the projection onto the vacuum state $\Omega \equiv 1$)

$$\left( (1 - P_{11})L, \mathcal{T}^0|v^0 - u^0| - 1 \mathcal{T}^0|u^0 - v^0| (1 - P_{11})L \right)_{\mathcal{H}} = \mathcal{G}(u, v), \ u^0 \neq v^0$$

where, with $M = (1 - P_{11})L$,

$$\mathcal{G}(u, v) = \mathcal{G}_{MM}(u, v)\chi_{u^0 \leq v^0} + \mathcal{G}_{MM}(u, v)\chi_{u^0 > v^0}
= \mathcal{G}_{MM}(u, v)\chi_{u^0 \leq v^0} + \mathcal{G}_{MM}^*(u, v)\chi_{u^0 > v^0},$$

and we have used the notation $z_\ell = (-z^0, \vec{z})$ if $z = (z^0, \vec{z})$.

$M$ may have contributions to the energy spectrum in the interval $(0, -(4 - \epsilon)\ln \kappa)$ that arise from states not in $\mathcal{H}_M$. We show this is not the case by considering the decay of the subtracted function

$$\mathcal{F} = \mathcal{G} - P\Lambda\mathcal{Q}$$

(60)

where the kernels of $P$, $\Lambda$ and $\mathcal{Q}$ are given by

$$P(u, w) = \mathcal{G}_{MM}(u, w)\chi_{u^0 \leq w^0} + \mathcal{G}_{MM}(u, w)\chi_{u^0 > w^0}
= \mathcal{G}_{MM}(u, w)\chi_{u^0 \leq w^0} + \mathcal{G}_{MM}^*(u, w)\chi_{u^0 > w^0},$$

$$\mathcal{Q}(z, v) = \mathcal{G}_{MM}(z, v)\chi_{z^0 \leq v^0} + \mathcal{G}_{MM}(z, v)\chi_{z^0 > v^0}
= \mathcal{G}_{MM}(z, v)\chi_{z^0 \leq v^0} + \mathcal{G}_{MM}^*(z, v)\chi_{z^0 > v^0},$$

$$\Lambda(u, v) = \mathcal{G}_{MM}(u, v)\chi_{u^0 \leq v^0} + \mathcal{G}_{MM}(u, v)\chi_{u^0 > v^0}$$
\[ \mathcal{J}(w, z) = G_{MM}(w, z) + \mathcal{G}_{MM}(w, z) \]
\[ = G_{MM}(w, z) \chi_{w^0 \leq z^0} + \mathcal{G}_{MM}(w, z) \chi_{w^0 > z^0} \]

with \( \Lambda(w, z) = \mathcal{J}^{-1}(w, z) \). The identities above are obtained using time reversal which gives

\[ G_{MM}(u_1, u_2) = G_{MM}^*(u_1, u_2), \quad G_{MM}(u, w) = G_{MM}^*(u, w), \]
\[ G_{MM}(z_1, z_2) = G_{MM}^*(z_1, z_2), \quad G_{MM}(w, z) = G_{MM}^*(w, z). \]

The motivation for the definitions of the kernels of \( \mathcal{P}, \Lambda \) and \( Q \) is such that time reflected points give the same value for the \( u^0 < v^0 \) and \( u^0 > v^0 \) definitions.

The kernels of \( \mathcal{P} \) and \( Q \) also have spectral representations for non-coincident temporal points given by

\[ \left( \mathcal{M}, \int^{u^0 - v^0}_{0} - 1 \right) \mathcal{M} = \mathcal{P}(u, v), \quad u^0 \neq v^0 \]
\[ \left( \mathcal{M}, \int^{u^0 - v^0}_{0} - 1 \right) \mathcal{M} = Q(u, v), \quad u^0 \neq v^0. \]

We remark that in the two equations above we made use of \( \langle \mathcal{M}(u) \rangle = \langle \mathcal{M}(u) \rangle = 0 \) by parity symmetry. Using the hyperplane decoupling method we show below that \( \mathcal{F}^{(r)}(u, v) = 0 \) for \( r = 0, 1, 2, 3 \) for \( |u^0 - v^0| > 2 \) which implies that \( \mathcal{F}(p) \) is analytic in \( p^0 \) in the strip \( |\text{Im} p^0| < (4 - \epsilon) \ln \kappa \). Again only the expansion in \( \kappa^p \) is needed because of our restriction \( \beta < \kappa \). But \( \mathcal{F}(p) = \tilde{\mathcal{G}}(p) - \tilde{\mathcal{P}}(p) \lambda(p) \tilde{Q}(p) \) so that possible singularities of \( \tilde{\mathcal{G}}(p) \) in the strip are cancelled by those in the term \( \tilde{\mathcal{P}}(p) \lambda(p) \tilde{Q}(p) \). From their spectral representations it is seen that \( \tilde{\mathcal{P}}(p) \) and \( \tilde{\mathcal{Q}}(p) \) only have singularities at the one-meson particle spectrum and the same holds for \( \tilde{\mathcal{P}}(p) \lambda(p) \tilde{Q}(p) \) since \( \lambda(p) \) is analytic in the strip. Thus the singularities of \( \tilde{\mathcal{G}}(p) \) and the spectrum generated by \( L \) in the interval \( (0, -(4 - \epsilon) \ln \kappa) \) are contained in the one-meson spectrum.

Expanding \( \mathcal{F} \) in Eq. (60) in powers of \( \kappa_p \) we get the result

\[ \mathcal{F} = \mathcal{F}^{(0)} \kappa^0_p + \mathcal{F}^{(1)} \kappa^1_p + \mathcal{F}^{(2)} \kappa^2_p + O(\kappa^3_p) \]
\[ = \left( \mathcal{G}^{(0)} - \mathcal{P}^{(0)} \Lambda^{(0)} Q^{(0)} \right) \kappa^0_p \]
\[ + \left( \mathcal{G}^{(1)} - \mathcal{P}^{(1)} \Lambda^{(0)} Q^{(0)} - \mathcal{P}^{(0)} \Lambda^{(1)} Q^{(0)} - \mathcal{P}^{(0)} \Lambda^{(0)} Q^{(1)} \right) \kappa^1_p \]
\[ + \left( \mathcal{G}^{(2)} - \mathcal{P}^{(2)} \Lambda^{(0)} Q^{(0)} - \mathcal{P}^{(0)} \Lambda^{(2)} Q^{(0)} - \mathcal{P}^{(0)} \Lambda^{(0)} Q^{(2)} \right) \kappa^2_p \]
\[ - \mathcal{P}^{(1)} \Lambda^{(1)} Q^{(0)} - \mathcal{P}^{(1)} \Lambda^{(0)} Q^{(1)} - \mathcal{P}^{(0)} \Lambda^{(1)} Q^{(1)} \right) \kappa^2_p + O(\kappa^3_p). \]

That \( \mathcal{F}^{(r)}(u, v) = 0 \) \( (r = 0, 1, 3) \) follows from gauge integration and imbalance of fermion fields appearing in the expectations. The second derivative of \( \mathcal{F}(u, v) \) for the time ordering \( u^0 < p < v^0 \) is

\[ \mathcal{F}^{(2)} = \mathcal{G}^{(2)} - \mathcal{P}^{(0)} \Lambda^{(0)} Q^{(2)} + \mathcal{P}^{(0)} \Lambda^{(0)} Q^{(2)} + \mathcal{P}^{(0)} \Lambda^{(0)} Q^{(0)} \]
\[ = A_1 + A_2 + A_3 + A_4. \]

We will use in the sequel, for \( r^0 \leq p < s^0 \) the following special cases of Eq. (14)

\[ G_{MM}^{(2)}(r, s) = \left[ G_{MM}^{(0)} \circ G_{MM}^{(0)} \right](r, s), \quad G_{MM}^{(2)}(r, s) = \left[ G_{MM}^{(0)} \circ G_{MM}^{(0)} \right](r, s) \]
\[ G_{MM}^{(2)}(r, s) = \left[ G_{MM}^{(0)} \circ G_{MM}^{(0)} \right](r, s), \quad G_{MM}^{(2)}(r, s) = \left[ G_{MM}^{(0)} \circ G_{MM}^{(0)} \right](r, s). \]

For the term \( A_2 \) we have

\[ A_2 = - \sum_{w^0 \leq z^0} \mathcal{P}^{(0)}(u, w) \Lambda^{(0)}(w, z) \left[ G_{MM}^{(0)} \circ G_{MM}^{(0)} \right](z, v) \]
\[ = - \sum_{w^0 \leq z^0} \mathcal{P}^{(0)}(u, w) \Gamma_{MM}^{(0)}((p + 1, w), v) \]
\[ = - \left[ G_{MM}^{(0)} \circ G_{MM}^{(0)} \right](u, v) = - A_1. \]
where in the equation above we have extended the sum to all $z$ using the support properties of $\Lambda^{(0)}$ and $G^{(0)}$.

For the term $A_4$ we get similarly $A_4 = A_2$. Now we consider the term $A_3$:

$$A_3 = - \sum_{w^0 \leq p, z^0 \geq p+1} \mathcal{P}^{(0)}(u, w)\Lambda^{(2)}(w, z)Q^{(0)}(z, v).$$

But for, $w^0 \leq p$, $z^0 \geq p + 1$, $\Lambda^{(2)}(w, z) = -[\Lambda^{(0)}(\mathcal{J}^{(2)}(w, z)) - \Lambda^{(0)}(w, z)](w, z)$ which is obtained taking the second derivative of the relation $A\mathcal{J} = 1$ and observing that $[\Lambda^{(0)}(\mathcal{J}^{(1)}(w, z)) - \Lambda^{(0)}(w, z)] = 0$ for $w^0 \leq p$, $z^0 \geq p + 1$. With these restrictions on sums we get

$$\mathcal{J}^{(2)}(x, y) = G^{(2)}_{\Lambda M M}(x, y) = \left[G^{(0)}_{\Lambda M M} \ast G^{(0)}_{\Lambda M M}\right](x, y)$$

so that

$$A_3 = \sum_{w^0, z^0 \leq p, x^0, y^0 \geq p + 1} \mathcal{P}^{(0)}(u, w)\Lambda^{(0)}(w, x) \left[G^{(0)}_{\Lambda M M} \ast G^{(0)}_{\Lambda M M}\right](x, y)\Lambda^{(0)}(y, z)Q^{(0)}(z, v).$$

Extending the sum to all $x^0$ and $y^0$ we get

$$A_3 = \sum_{w} \mathcal{P}^{(0)}(u, (p, \vec{w}))(p + 1, \vec{w}), v) = \left[G^{(0)}_{\Lambda M M} \ast G^{(0)}_{\Lambda M M}\right](u, v) = A_1.$$

Collecting the results above we have $\mathcal{J}^{(2)}(u, v) = 0$ for $w^0 \leq p < v^0$.

The treatment for the other time ordering is more intricate, but similar. For $w^0 \geq p > v^0$ we have:

$$\mathcal{J}^{(2)} = G^{(2)} - \mathcal{P}^{(0)}(u, w)\Lambda^{(0)}Q^{(2)} - \mathcal{P}^{(0)}(u, w)\Lambda^{(2)}Q^{(0)} - \mathcal{P}^{(2)}(u, w)Q^{(0)} = A'_1 + A'_2 + A'_3 + A'_4. \tag{61}$$

Considering the $A'_2$ term in the expression above we get:

$$A'_2 = - \sum_{w^0, z^0 > p} \mathcal{P}^{(0)}(u, w)\Lambda^{(0)}(w, z) \left[G^{(0)}_{\Lambda M M} \ast G^{(0)}_{\Lambda M M}\right](x, v)$$

$$= - \sum_{w^0, z^0 > p} \mathcal{P}^{(0)}(u, w)\Lambda^{(0)}(w, z)G^{(0)}_{\Lambda M M}(z, (p + 1, \vec{w}))G^{(0)}_{\Lambda M M}(p, \vec{w}, v). \tag{62}$$

We write

$$G^{(0)}_{\Lambda M M}(z, (p + 1, \vec{w})) = G^{(0)}_{\Lambda M M}(z, (p + 1, \vec{w}))\chi_{z^0 \geq p + 1} + G^{(0)}_{\Lambda M M}(z, (p + 1, \vec{w}))\chi_{z^0 \leq p + 1}$$

$$+ G^{(0)}_{\Lambda M M}(z, (p + 1, \vec{w}))\delta_{z^0, p + 1} - G^{(0)}_{\Lambda M M}(z, (p + 1, \vec{w}))\delta_{z^0, p + 1} \tag{63}$$

under the $w^0, z^0$ summations.

Recall that

$$\mathcal{J}(z, (p + 1, \vec{w})) = G_{\Lambda M M}(z, (p + 1, \vec{w}))\chi_{z^0 \leq p + 1} + G_{\Lambda M M}(z, (p + 1, \vec{w}))\chi_{z^0 > p + 1}.$$  

For the first two terms in Eq. (63), going back to Eq. (62), we have:

$$-\mathcal{P}^{(0)}(u, x)\Lambda^{(0)}(x, y)(A'_5 + A'_6)(y, z)G^{\ast}_{\Lambda M M}(z, v)$$

$$= - \sum_{w} \mathcal{P}^{(0)}(u, w)\Lambda^{(0)}(w, z)\mathcal{J}^{(0)}(z, (p + 1, \vec{w}))G^{\ast}_{\Lambda M M}((p, \vec{w}), v)$$

$$= - \sum_{w} \mathcal{P}^{(0)}(u, w)(\delta_{w^0, p + 1})G^{\ast}_{\Lambda M M}((p, \vec{w}), v)$$

$$= - \sum_{w} G^{(0)*}_{\Lambda M M}(u, (p + 1, \vec{w}))G^{\ast}_{\Lambda M M}((p, \vec{w}), v)$$

$$= - \sum_{w} G^{(0)*}_{\Lambda M M} \circ G^{\ast}_{\Lambda M M}(u, v) = -A'_1.$$
for $u^0 > p + 1$. For the remaining terms we get that

$$A'_2 + A'_3 = G^{(0)*}_{\mathcal{M},\mathcal{M}}(z, (p + 1, w)^0) \delta_{w^0, p+1} - G^{(0)}_{\mathcal{M},\mathcal{M}}(z, (p + 1, w)^0) \delta_{w^0, p+1},$$

which is zero by time reversal symmetry $T$.

For the term $A'_3$ we have:

$$A'_3 = - \sum_{w^0 > p, z^0 \geq p} P^{(0)}(u, w) \Lambda^{(2)}(w, z) Q^{(0)}(z, v)$$

with this $w_0, z_0$ restrictions

$$\Lambda^{(2)}(w, z) = - \sum_{x^0 > p, y^0 \geq p} \Lambda^{(0)}(w, x) J^{(2)}(x, y) \Lambda^{(0)}(y, z)$$

and with the $x_0, y_0$ restrictions we have

$$J^{(2)}(x, y) = G^{(2)*}_{\mathcal{M},\mathcal{M}}(x, y) = \left[ G^{(0)*}_{\mathcal{M},\mathcal{M}} \circ G^{(0)}_{\mathcal{M},\mathcal{M}} \right](x, y).$$

Thus

$$A'_3 = \sum_{w^0, x^0 > p, z^0, y^0 \leq p} P^{(0)}(u, w) \Lambda^{(0)}(w, x) \left[ G^{(0)*}_{\mathcal{M},\mathcal{M}} \circ G^{(0)}_{\mathcal{M},\mathcal{M}} \right](x, y) \Lambda^{(0)}(y, z) Q^{(0)}(z, v).$$

Considering $x, y$ sums in the second and fifth factors above we get, $r^0 = p + 1$,

$$A'_3 = \sum_{x^0 > p, y^0 \leq p} \Lambda^{(0)}(w, x) \left[ G^{(0)*}_{\mathcal{M},\mathcal{M}}(x, r) \chi_{x^0 > p+1} + G^{(0)}_{\mathcal{M},\mathcal{M}}(x, r) \chi_{x^0 \leq p+1} \right]$$

$$+ G^{(0)*}_{\mathcal{M},\mathcal{M}}(x, r) \delta_{x^0, p+1} - G^{(0)}_{\mathcal{M},\mathcal{M}}(x, r) \delta_{x^0, p+1} \right] G^{(0)*}_{\mathcal{M},\mathcal{M}}(r, y) \Lambda^{(0)}(y, z).$$

The $x$ sum for $x^0 = p + 1$, which upon using Eq. (63), gives

$$\delta_{w^0, p+1} + \sum_{x} \Lambda^{(0)}(w, x) \left[ G^{(0)*}_{\mathcal{M},\mathcal{M}}(x, (p + 1, r)) - G^{(0)}_{\mathcal{M},\mathcal{M}}(x, (p + 1, r)) \right]$$

and using time reversal the term in $\left[ \cdot \right]$ is zero. Similarly the sum in $y$ gives $\delta_{p, x^0}$.

Returning to $A'_3$ we have for $u^0 > p + 1, v^0 < p$:

$$A'_3 = \sum_{(p, \bar{w})} P^{(0)}(u, (p + 1, \bar{w})) Q^{(0)*}(u, v)$$

$$= \sum_{(p, \bar{w})} G^{(0)*}_{\mathcal{M},\mathcal{M}}(u, (p + 1, \bar{w})) G^{(0)*}_{\mathcal{M},\mathcal{M}}(u, v) = \left[ G^{(0)*}_{\mathcal{M},\mathcal{M}} \circ G^{(0)}_{\mathcal{M},\mathcal{M}} \right](u, v) = A'_1.$$

Similarly, we have $A'_4 = A'_2$. Thus $F^{(2)}(u, v) = 0$ for $u^0 > p + 1$ or $v^0 < p$ so we have a minimum separation of $|u^0 - v^0| > 2$ to get that $F^{(2)}(u, v) = 0$ and we are done.

VI. CONCLUDING REMARKS

We completed the exact determination of the one-particle E-M spectrum associated with the 3 + 1 Wilson’s lattice QCD model with three quark flavors initiated in Refs. [20, 21] for the odd sector $H_o$ of the physical Hilbert space, where the baryons (of asymptotic mass $-3 \ln \kappa$) lie. Here we analyzed the even sector $H_e$ and obtained the eightfold way mesons (of asymptotic mass $-2 \ln \kappa$) from first principles, i.e. directly from the quark gluon dynamics. We obtain a spectral representation for the two-point correlation. The non-singular part of the eightfold way meson masses are given by a function jointly analytic in $\kappa$ and $\beta$. In this way, in particular, we control the expansion of the mass to all orders in $\kappa$ and $\beta$. We obtain a pseudo-scalar vector meson mass splitting given by $2\kappa^4 + O(\kappa^6)$ at $\beta = 0$ and, by
analyticity, the splitting persists for $\beta > 0$, $\beta < < \kappa$. A correlation subtraction method is used to guarantee that there is no other spectrum in all $\mathcal{H}_c$ except that generated by the eightfold way mesons up to near the two-meson threshold ($\approx -4 \ln \kappa$). Combining this result with a similar one for baryons in Refs. [20, 21], the one-hadron E-M spectrum in all $\mathcal{H} \equiv \mathcal{H}_c \oplus \mathcal{H}_a$, up to near the two-meson threshold, is the one generated by the Gell-Mann and Ne’eman eightfold gauge-invariant meson and baryon fields. Thus, confinement is proved up to near the two-meson threshold.

The determination of the one-meson spectrum is an essential step towards the analysis of the existence of two-hadron bound states as we did previously in simpler QCD models. Hence, our present work opens the way to attack interesting open questions such as the existence of tetra-quark and pentaquark states, for example, meson-meson and meson-baryon bound states.

VII. ACKNOWLEDGEMENTS

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APPENDIX A: SYMMETRY CONSIDERATIONS

Now we list several symmetries used to obtain the general structure of Eq. (49) and the relations among $\mathcal{G}$’s for distinct lattice points. Those properties are used to simplify the proof of Theorem 3 devoted to Appendix B. Before we list the symmetries and determine the properties of the two-point function of Eq. (16) we remark that the use of $g_p$ plays a fundamental role in our analysis. The $(4 \times 4)$ matrix in the total spin basis ($G_{\mathcal{J}, \mathcal{P}}$) can be reduced to an even more diagonal form. More precisely, it breaks into a direct sum with one block (1 $G$ of distinct lattice points. Those properties are used to simplify the proof of Theorem 3 devoted to Appendix B. Before except for time reversal where the transformed field equals the complex conjugate of the field average.

Now we list several symmetries used to obtain the general structure of Eq. (49) and the relations among $\mathcal{G}$’s.

- Time reversal $T$: $\psi_\alpha(x) \rightarrow \bar{\psi}_\beta(x_t) A_{\alpha \beta}, \bar{\psi}_\alpha(x) \rightarrow B_{\alpha \beta} \bar{\psi}(x_t), A = B = B^{-1} = g^0, f(g_{xy}) \rightarrow f^*(g_{x,y_t})$, with $z_t = (-z^0, \bar{z})$.

- Parity $P$: $\psi_\alpha(x) \rightarrow A_{\alpha \beta} \psi_\beta(x_p), \bar{\psi}_\alpha(x) \rightarrow \bar{\psi}_\beta(x_p) B_{\alpha \beta}, A = B = B^{-1} = g^0, f(g_{xy}) \rightarrow f(g_{x,y_p}),$ with $z_p = (z^0, -\bar{z})$.

- Charge conjugation $C$: $\psi_\alpha(x) \rightarrow \bar{\psi}_\beta(x) A_{\alpha \beta}, \bar{\psi}_\alpha(x) \rightarrow B_{\alpha \beta} \bar{\psi}(x), A = -B = B^{-1} = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, f(g_{xy}) \rightarrow f(g_{x,y})$.

- Time Reflection $T$: $\psi_\alpha(x) \rightarrow A_{\alpha \beta} \bar{\psi}(x), \bar{\psi}_\alpha(x_1) \rightarrow \bar{\psi}_\beta(x_1) B_{\alpha \beta}$, where $B = A^{-1} = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}$, $f(g_{xy}) \rightarrow f(g_{x,y_t})$, recall that $z_t = (-z^0, \bar{z})$.

- Reflection in $c^1$: $\psi_\alpha(x) \rightarrow A_{\alpha \beta} \bar{\psi}(x_r), \bar{\psi}_\alpha \rightarrow \bar{\psi}_\beta(x_r) B_{\alpha \beta}$ where $A = B^{-1} = \text{diag}(e^{-i\theta}, e^{i\theta}, e^{-i\theta}, e^{i\theta}), f(g_{xy}) \rightarrow f(g_{x,y_r}),$ with $z_r = (z^0, -z^2, z^1, z^3)$ and $\theta = \pi/4$.

The above symmetries are defined on single fields, extended linearly to polynomials and taken to be order preserving, except for $T$, which is anti-linear, and $C$, both of which are order reversing. For all of them the action is invariant, the transformed fields equals the field average, except for time reversal where the transformed field equals the complex conjugate of the field average.

We are now ready to state the following Lemma:

Lemma 1 The following properties of symmetry holds for $G$, $\Lambda$, $G$ and $\Gamma$.

1. $G_{\alpha \beta}(x) = G_{\beta \alpha}^*(x)$;
2. $G_{\alpha \beta}(x_t) = G_{\beta \alpha}^*(x) = G_{\alpha \beta}(x)$;
3. $G_{\alpha \alpha}(x) = G_{\alpha \alpha}^*(x_t) = G_{\alpha \alpha}^*(x)$; and the same for $G$, $\Lambda$ and $\Gamma$. 

4. Using the ordering $\alpha = 1 = (3,1)$, $\alpha = 2 = (4,2)$, $\alpha = 3 = (4,1)$ and $\alpha = 4 = (3,2)$, for fixed isospin and hypercharge, $(G_{\alpha \beta})$ has the following structure, with $a, d \in \mathbb{R}$, and $b, c, e \in \mathbb{C}$

\[
(G_{\alpha \beta}) = \begin{pmatrix}
a & b & c & e \\
b^* & a & -c & -e^* \\
\bar{c} & -c^* & d & e \\
\bar{e} & -e^* & d & \bar{e} \\
\end{pmatrix}; \tag{A1}
\]

hence the $(4 \times 4)$ matrix $(G_{\mathcal{J} \mathcal{J}'})$ in the total spin basis has the following structure

\[
(G_{\mathcal{J} \mathcal{J}'}) = \begin{pmatrix}
a & 0 & 0 & 0 \\
ob & c & d & 0 \\
0 & c^* & e & 0 \\
0 & 0 & \bar{c}^* & \bar{e} \\
\end{pmatrix}; \tag{A2}
\]

and the same structure holds for $(\Gamma_{\mathcal{J} \mathcal{J}'})$.

5. For $\chi \in \mathbb{R}$, let $p^0 = i \chi$. We obtain $\tilde{G}_{\mathcal{J} \mathcal{J}'}(i \chi, \vec{p}) = \tilde{G}^*_{\mathcal{J} \mathcal{J}'}(i \chi, \vec{p})$;

6. When $\vec{p} = \vec{0}$, $\tilde{G}_{\mathcal{J} \mathcal{J}'}(p^0, \vec{p} = \vec{0}) = \text{diag}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$, $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{R}$;

and the same for the matrix $\Gamma$.

**Proof:** Items 1, 2, and 3 follow directly by applying parity and time reversal symmetries. To prove item 4, we apply the spin flip symmetry $\mathcal{F}_\pi = -i T CT$ and use the fact that the matrix $(G_{\alpha \beta})$ is self-adjoint as follows from previous items. Hence, in the individual spin basis we get the structure

\[
(G_{\alpha \beta}) = \begin{pmatrix}
a & d & c & e \\
d^* & a & -e^* & -c \\
\bar{c} & -c^* & b & f \\
\bar{e} & -e^* & \bar{f} & \bar{b} \\
\end{pmatrix}.
\]

Next, we use charge conjugation followed by permutation of isospin indices. Explicitly, for the block associated with $\mathcal{M}_3^u$ or $\mathcal{M}_3^d$ we consider the permutation matrix, acting on the isospin degree of freedom, $U \equiv P_{ud} \in SU(3)_f$ which interchanges $(u \rightleftarrows d)$ and is given by Eq. (41). By the global flavor symmetry we get, suppressing all but the isospin index,

\[
\langle(U^\dagger \psi)_{J_1} (\bar{\psi} U)_{J_2} (\bar{\psi} U)_{J_3} (\psi)_{J_4} \rangle = \langle \psi_{J_1} \bar{\psi}_{J_2} \bar{\psi}_{J_3} \psi_{J_4} \rangle \tag{A3}
\]

from which $c = e^*$, and hence, Eq. (A1) follows. The treatment for the other blocks is similar, except by the flavor permutation matrix which, according to the isospin indices of each block, interchanges $(d \rightleftarrows s)$ and $(u \rightleftarrows s)$. We now pass to the total spin basis which is related to the individual spin basis by the matrix $A$ of Eq. (44), i.e. $(G_{\mathcal{J} \mathcal{J}'}) = A(G_{\alpha \beta})A^T$ and we get Eq. (A2).

The prove item 5, recalling that

\[
\tilde{G}_{\mathcal{J} \mathcal{J}'}(i \chi, \vec{p}) = \sum_{x^0, \vec{x}} e^{iq_0} e^{-i\bar{q} \cdot \vec{x}} G_{\mathcal{J} \mathcal{J}'}(x^0, \vec{x}),
\]

we obtain, by the first Item of this Lemma,

\[
\tilde{G}_{\mathcal{J} \mathcal{J}'}(i \chi, \vec{p}) = \sum_{x^0, \vec{x}} e^{iq_0} e^{-i\bar{q} \cdot \vec{x}} G^*_{\mathcal{J} \mathcal{J}'}(x^0, \vec{x})
\]

\[
= \left( \sum_{x^0, \vec{x}} e^{iq_0} e^{i\bar{q} \cdot \vec{x}} G_{\mathcal{J} \mathcal{J}'}(x^0, \vec{x}) \right)^* = \tilde{G}^*_{\mathcal{J} \mathcal{J}'}(i \chi, -\vec{p}),
\]

and, using parity symmetry, $G_{\mathcal{J} \mathcal{J}'}(x^0, \vec{x}) = G_{\mathcal{J} \mathcal{J}'}(x^0, -\vec{x})$ which implies that $\tilde{G}_{\mathcal{J} \mathcal{J}'}(p^0, \vec{p}) = \tilde{G}_{\mathcal{J} \mathcal{J}'}(p^0, -\vec{p})$, and the proof of the fifth item follows.

Finally, the proof of item 6 follows using $\pi/2$ rotations about $e^3$.

Using Lemma 1 and Lemma 2 below, we only need to prove Theorem 3 for $x^0 > 0$ and $\epsilon, \epsilon', \epsilon'' = 1$, for $ij = 12, 13.$
Lemma 2 For $\rho, \sigma = 0, 1$ and $\epsilon, \epsilon' \in \{-1, +1\}$, the following relations are verified:

1. $G_{aa}(0, \rho \epsilon^0 + \epsilon \epsilon') = G_{aa}(0, \rho \epsilon^0 + \epsilon + \sigma \epsilon')$.
2. $G_{aa}(0, \rho \epsilon^0 + \epsilon + \sigma \epsilon^3) = G_{aa}(0, \rho \epsilon^0 + \epsilon^2 + \sigma \epsilon^3)$.
3. $G_{12}(0, \rho \epsilon^0 + \epsilon + \sigma \epsilon^3) = -G_{34}(0, \rho \epsilon^0 + \epsilon^2 + \sigma \epsilon^3)$.

Proof: Items 1, 2, and 3 all follow using rotation of $\pi/2$ about $\epsilon^3$, reflections about $\epsilon^3$ and parity.

APPENDIX B: SMALL DISTANCE BEHAVIOR OF $G$ AND $\Lambda$

In this Appendix, we consider the contributions obtained expanding $G(0, x)$ in powers of $\kappa$, but all we need are non-intersecting paths. In particular, we develop a general formula with applications to the determination of the small distance behavior of $G$ and $\Lambda$. Recall that $\Lambda$ is defined as a Neumann series according to Eq. (25), and we will need the short distance behavior of $\Lambda$ until and including $O(G_n^5)$,

$$\Lambda = \sum_{i=0}^{5} (-1)^{i}[G_n]^i + O(G_n^6),$$

(B1)

where we made use of $G_{0}^{-1} = 1 + O(\kappa^8)$. When $x \neq 0$, $G(0, x) = G_n(0, x)$ and Eq. (B6) furnishes us with a formula to determine $G_n(0, x)$ in Eq. (B1) for nonintersecting paths.

In Theorems 2 and 3 we will use gauge integrals with two overlapping bonds of opposite orientation given by

$$I_2 = \int U_{a_1 b_1}(g)U_{a_2 b_2}^{-1}(g)d\mu(g) = \int g_{a_1 b_1}g_{a_2 b_2}^{-1}d\mu(g) = \frac{1}{3} \delta_{a_1 b_2} \delta_{a_2 b_1}.$$  

(B2)

Although we won’t use the gauge integral for three bonds with the same orientation, namely $I_3$, in Theorems 2 and 3, we present it here since, $I_3$ was used in the determination of the coefficient $G_{LL}^{(0,3)}$ in Eq. (9),

$$I_3 = \int g_{a_1 b_1}g_{a_2 b_2}g_{a_3 b_3}d\mu(g) = \frac{1}{6} \epsilon_{a_1 a_2 a_3} \epsilon_{b_1 b_2 b_3}.$$  

(B3)

Also, we will need gauge integrals with four and six overlapping bonds [in which case we use the convenient notation (123) $\equiv ((a_1 b_1), (a_2 b_2), (a_3 b_3))$, (132) $\equiv ((a_1 b_1), (a_3 b_3), (a_2 b_2))$, etc]

$$I_4 = \int g_{a_1 b_1}g_{a_2 b_2}^{-1}g_{a_3 b_3}^{-1}g_{a_4 b_4}^{-1}d\mu(g)$$

$$= \frac{1}{5!} [\delta_{a_1 b_2} \delta_{a_3 b_4} \delta_{b_1 a_2} \delta_{b_3 a_4} + \{a_2 \leftarrow a_4; b_2 \leftarrow b_4\}]$$

$$+ \frac{1}{24} [\delta_{a_1 b_2} \delta_{a_3 b_4} \delta_{b_1 a_4} \delta_{b_3 a_2} + \{a_2 \leftarrow a_4; b_2 \leftarrow b_4\}]$$

(B4)

$$I_6 = \int g_{a_1 b_1}g_{a_2 b_2}^{-1}g_{a_3 b_3}^{-1}g_{a_4 b_4}^{-1}g_{a_5 b_5}^{-1}g_{a_6 b_6}^{-1}d\mu(g)$$

$$= \frac{1}{5!} \epsilon_{a_1 a_2 a_3} \epsilon_{b_1 b_2 b_3} \epsilon_{a_4 a_5 a_6} \epsilon_{b_4 b_5 b_6} + \frac{1}{24} [\delta_{a_1 b_2} \delta_{a_3 b_4} \delta_{a_5 b_6} \delta_{a_2 b_3} \delta_{a_4 b_5} \delta_{a_6 b_1} + \{123 \rightarrow 132\}$$

$$+ \{123 \rightarrow 231\} + \{123 \rightarrow 321\} + \{123 \rightarrow 321\}] + \{123 \rightarrow 132\}$$

(B5)

In non-intersecting paths only $I_2$ occurs. The general formula for non-intersecting path is given by

$$\langle M_{\alpha f}(0)M_{\beta \tilde{f}}(x) \rangle =_p \frac{\kappa^L}{2} \delta_{\tilde{f}f} \Gamma_{\alpha \beta} \Gamma_{\alpha \beta}^{-p} \Gamma_{\alpha \alpha}$$

(B6)

where we recall that $\tilde{\alpha} = (\alpha, \alpha, u)$, $\tilde{\beta} = (\beta, \beta, u)$, $\tilde{f} = (f_1, f_2)$, $\tilde{p} = (f_3, f_4)$ and $L$ is the length of the path. The subscript $p$ in Eq. (B6) above means that we take only the contribution coming from nonintersecting paths, with any consecutive points of the path linked by two overlapping bonds of opposite orientation.
The notation $\Gamma^\rho_{\alpha\beta}$ $(\Gamma^{-\rho}_{\alpha\beta})$ means the $\alpha \beta$ element of the ordered product of $\Gamma$ matrices along the path that connects 0 to $x$ (to 0). For example, if $x = e^0 + e^1 + e^2$ and the path is chosen such that $0 \rightarrow e^0 \rightarrow e^0 + e^1 \rightarrow e^0 + e^1 + e^2$, hence $\Gamma^\rho = \Gamma^0 \Gamma^1 \Gamma^2$ where we have used the notation $\Gamma^{\rho \epsilon} \equiv \Gamma^\rho \Gamma^\epsilon$. On the other way, for the reversing path, the product of the $\Gamma$ matrices is in the opposite order, i.e. $\Gamma^{-\rho} = \Gamma^{-2} \Gamma^{-1} \Gamma^{-0}$. In general, if the path goes as follows $0 \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_n \rightarrow x$ then $\Gamma^\rho = \Gamma^{0-x} = \Gamma^{x_1} \Gamma^{x_2-x_1} \ldots \Gamma^{x_n-x_{n-1}} \Gamma^{x-x_n}$ and $L = n + 1$. In the same way we have $\Gamma^{-\rho} = \Gamma^{x-x_0} = \Gamma^{-(x-x_n)} \Gamma^{-(x-x_{n-1})} \ldots \Gamma^{-(x_2-x_1)} \Gamma^{x-x_1}$.

We give a brief deduction of the general formula (B6). Expanding the exponential of the action in the numerator of $\langle \mathcal{M}_{\tilde{\alpha} \tilde{f}}(0) \mathcal{M}_{\tilde{\beta} \tilde{p}}(x) \rangle$ we pick up two overlapping bonds with opposite orientation for each bond of the path. Using $I_2$ and carrying out the fermi integration over the intermediate fields we arrive at two kinds of products of $\Gamma$ matrices, one of them is zero using the come and go property i.e., $\Gamma^\rho \Gamma^{-\rho} = 0$. For the remaining product we get

$$\langle \mathcal{M}_{\tilde{\alpha} \tilde{f}}(0) \mathcal{M}_{\tilde{\beta} \tilde{p}}(x) \rangle = p \left( \frac{\kappa}{2} \right)^{2L} \langle \psi_{a,\alpha_1,f_1} \bar{\psi}_{a,\alpha_2,f_2} \bar{\psi}_{a,\alpha_3,g_1} \psi_{a,\alpha_4,g_2} \rangle \langle \psi_{a,\beta_1,g_1} \bar{\psi}_{a,\beta_2,g_2} \bar{\psi}_{a,\beta_3,f_1} \psi_{a,\beta_4,f_2} \rangle \langle \Gamma^\rho_{\alpha_1,\beta_1} \Gamma^{-\rho}_{\alpha_2,\beta_2} \rangle$$

where $\langle \cdot \rangle^{(0)}$ is the expectation with the hopping parameter $\kappa$ set to zero in the action $\mathcal{S}$. Note that the expectations in Eq. (B7) can be easily calculated and the result is

$$\langle \psi_{a,\alpha_1,f_1} \bar{\psi}_{a,\alpha_2,f_2} \bar{\psi}_{a,\alpha_3,g_1} \psi_{a,\alpha_4,g_2} \rangle^{(0)}/3 = (\delta_{\alpha_1,\alpha_2} \delta_{\alpha_3,\alpha_4} - \delta_{\alpha_1,\alpha_4} \delta_{\alpha_3,\alpha_2}) \delta_{f_1} \delta_{f_2} \delta_{g_1} \delta_{g_2}$$

and

$$\langle \psi_{a,\alpha_1,f_1} \bar{\psi}_{a,\alpha_2,f_2} \bar{\psi}_{a,\alpha_3,g_1} \psi_{a,\alpha_4,g_2} \rangle^{(0)}/3 = (\delta_{\alpha_1,\alpha_4} \delta_{\alpha_3,\alpha_2} - \delta_{\alpha_1,\alpha_2} \delta_{\alpha_3,\alpha_4}) \delta_{f_1} \delta_{f_2} \delta_{g_1} \delta_{g_2}$$

where we made use of the following notation, for the matrices $(\delta_{\tilde{a} \tilde{a}^\prime})$ and $(\delta_{\tilde{a} \tilde{a}^\prime})$.

$$(\delta_{\tilde{a} \tilde{a}^\prime}) = \begin{pmatrix} \delta_{\alpha_1,\alpha_1} & \delta_{\alpha_1,\alpha_2} \\ \delta_{\alpha_2,\alpha_1} & \delta_{\alpha_2,\alpha_2} \end{pmatrix}, \quad (\delta_{\tilde{a} \tilde{a}^\prime}) = \begin{pmatrix} \delta_{\beta_1,\beta_1} & \delta_{\beta_1,\beta_2} \\ \delta_{\beta_2,\beta_1} & \delta_{\beta_2,\beta_2} \end{pmatrix}.$$ (B10)

From Eq. (B7), (B8) and (B9) and carrying out the sum over $\tilde{g}$ we get

$$\langle \mathcal{M}_{\tilde{\alpha} \tilde{f}}(0) \mathcal{M}_{\tilde{\beta} \tilde{p}}(x) \rangle = p \left( \frac{\kappa}{2} \right)^{2L} \delta_{f_1} \delta_{f_2} \delta_{g_1} \delta_{g_2} \langle \Gamma^\rho_{\alpha_1,\beta_1} \Gamma^{-\rho}_{\alpha_2,\beta_2} \rangle$$

or more explicitly

$$\langle \mathcal{M}_{\tilde{\alpha} \tilde{f}}(0) \mathcal{M}_{\tilde{\beta} \tilde{p}}(x) \rangle = p \left( \frac{\kappa}{2} \right)^{2L} \delta_{f_1} \delta_{f_2} \delta_{g_1} \delta_{g_2} \langle \Gamma^\rho_{\alpha_1,\beta_1} \Gamma^{-\rho}_{\alpha_2,\beta_2} - \delta_{\alpha_1,\alpha_4} \delta_{\alpha_3,\alpha_2} \delta_{\alpha_2,\alpha_4} \delta_{\alpha_3,\alpha_1} \rangle.$$ (B12)

Formula (B6) follows from Eq. (B12) by observing that $\delta_{\alpha_1,\alpha_4} = \delta_{\beta_1,\beta_4} = 0$ since $\{\alpha_1,\beta_1\}$, $\{\alpha_4,\beta_4\}$ are lower and upper spin index sets, respectively.

Before we prove Theorem 3 we note that many possible configurations are shown to be zero using the come and go property of the $\Gamma$ matrices ($\rho = 0, 1, 2, 3, \epsilon = \pm 1$).

$$\Gamma^{\epsilon \epsilon^\prime} \Gamma^{-\epsilon \epsilon^\prime} = 0.$$ (B13)

For example, if a path doubles back upon itself at an isolated point then at this point the come and go property of Eq. (B13) holds to give a zero contribution. Also, possible contributions can give zero due to an imbalance of the number of fermions or the number of fermion components at a site.

Also, other useful properties involving $\Gamma$ matrices ($\rho = 0, 1, 2, 3, \epsilon = \pm 1$) used in evaluating possible contributions are

$$\Gamma^{\epsilon \epsilon^\prime} \Gamma^{\epsilon \epsilon^\prime} = -2 \Gamma^{\epsilon \epsilon^\prime}$$

$$\Gamma^{\epsilon \epsilon^\prime} \Gamma^{-\epsilon \epsilon^\prime} = 2 I_4 - \Gamma^{-\epsilon \epsilon^\prime} \Gamma^{\epsilon \epsilon^\prime}.$$ (B15)
Especially, the property in Eq. (B14) shows that lattice bond segments in a straight line behave similarly. Finally, the property of Eq. (B15) is useful to sum over different orders in a path with fixed endpoints. Due to Lemma 2 of Appendix A, we only need to consider the points \( x = 0, e^a, 2e^a, 3e^a, e^b, e^a + e^b, e^a + e^c, e^b + 2e^a, 2e^b + e^c \) with \( \rho = 0, 1 \) and \( \sigma = 2, 3 \) in the proof below.

In what follows, whenever we write \( x \rightarrow y \) connecting two distinct points on the lattice \( x \) and \( y \), \( \rightarrow \) means a link of the path, i.e. two opposite oriented bonds connecting the points \( x \) and \( y \).

We now turn to the proof of Theorem 3:

**Proof of Theorem 3 Item 1:** The proof of the short-distance behavior of \( \mathcal{G} \) follows directly from the non-intersecting path formula in Eq. (B6). We give details:

- We begin by considering \( x = 0 \). Using the symmetry of \( \pi/2 \) rotations about \( \rho \) shows that the off-diagonal elements are zero. The gauge integral and Eq. (B13) show that the first non-vanishing contribution occurs at \( \kappa^8 \), and consists of two paths which go around a square in opposite directions. The square has one vertex at zero.

- Take now \( x = \epsilon^1, \epsilon^6 \). We have a straightforward application of the path formula of Eq. (B6). The \( \kappa^4 \) and \( \kappa^6 \) contributions of the type \( 0 \rightarrow \epsilon^0 \rightarrow 0 \) and \( 0 \rightarrow \epsilon^0 \rightarrow 0 \rightarrow \epsilon^0 \), respectively, are zero by the come and go property of Eq. (B13). For \( x = \epsilon^6 \), parity symmetry at the level of correlations can also be used to show that the \( \kappa^4 \) contribution is zero. The non-vanishing \( \kappa^6 \) contribution come from paths of the type \( 0 \rightarrow \epsilon^j \rightarrow \epsilon^0 + \epsilon^j \rightarrow \epsilon^0 \), which we call U’s, meaning two oppositely oriented bonds on the three sides of the path \( 0 \rightarrow \epsilon^j \rightarrow \epsilon^0 + \epsilon^j \rightarrow \epsilon^0 \). A direct application of Eq. (B6) gives \( \mathcal{G}(\epsilon^0) = 3 \epsilon^0/8 \) and \( \mathcal{G}(\epsilon^1) = 3 \kappa^6/2 \).

- For \( x = \epsilon^1 + \epsilon^6 \), we have a straightforward application of the path formula. The same holds also for \( x = 2\epsilon^0, x = 2\epsilon^1, x = \epsilon^1 + \epsilon^2, x = \epsilon^1 + \epsilon^3, x = \epsilon^0 + 2\epsilon^1, x = 2\epsilon^0 + \epsilon^1, x = \epsilon^0 + \epsilon^1 + \epsilon^2, x = \epsilon^0 + \epsilon^1 + \epsilon^3, x = 3\epsilon^0 \). There are vertical (temporally oriented) \( \mathcal{U} \). The vertical contributions are given by the path formula as well as the \( \mathcal{U} \) contributions. Possible contributions which are vertical backtracking paths, i.e. the path \( 0 \rightarrow \epsilon^0 \rightarrow 2\epsilon^0 \rightarrow 0 \rightarrow 2\epsilon^0 \rightarrow 3\epsilon^0 \) are zero using imbalance of fermion components at \( \epsilon^0 \) or at \( 2\epsilon^0 \). For the points \( x = \epsilon^a + \epsilon^b \left( \epsilon^a \right) \) the contribution of \( \mathcal{O}(\kappa^6) \) is zero by the come and go property of Eq. (B13).

**Proof of Theorem 3 Item 2:** To prove this item we use the simplified formula for \( \Lambda \) as follows from Eq. (B1).

- For \( x = 0 \) we get by Eq. (B1) that \( \Lambda(0) = \mathcal{G}(0) - \mathcal{G}(0) + \mathcal{O}(\kappa^8) \) with \( \mathcal{G}(0) = \mathcal{O}(1) \) and \( \mathcal{G}(0) = \mathcal{O}(\kappa^4) \). Noting that \( \mathcal{G}(0) = \mathcal{G}(0) \mathcal{G}(x, 0) \) and taking into account contributions coming from \( x = 0, e^0, e^0, \epsilon^0 \), the result follows using Eq. (B6).

- For \( x = \epsilon^1 \) we get by Eq. (B1) that \( \Lambda(x) = \mathcal{G}(x) + \mathcal{O}(\kappa^6) \) with \( \mathcal{G}(x) = \mathcal{O}(\kappa^2) \). A direct application of Eq. (B6) gives the result.

- For \( x = \epsilon^6 \), the \( \kappa^2 \) contributions comes directly from Eq. (B6). The \( \kappa^4 \) contribution is zero due to property (B13). Finally, the \( \kappa^6 \) contribution is related to vertical paths, such as those connecting \( 0 \rightarrow \epsilon^0 \), i.e. (a) \( 0 \rightarrow \epsilon^0 \rightarrow 0 \rightarrow \epsilon^0 \); (b) \( 0 \rightarrow \epsilon^0 \rightarrow 2\epsilon^0 \rightarrow \epsilon^0 \); (c) \( 0 \rightarrow \epsilon^0 \rightarrow 0 \rightarrow \epsilon^0 \) contributing to \( \Lambda(\epsilon^0) \) as:

\[
\begin{align*}
(a) \quad & \mathcal{G}_n(\epsilon^0) = \mathcal{G}(0) \mathcal{G}(e^0, 0) \mathcal{G}(0, \epsilon^0); \\
(b) \quad & \mathcal{G}_n(\epsilon^0) = \mathcal{G}(0, 2e^0) \mathcal{G}(2e^0, \epsilon^0), \quad \mathcal{G}_n(\epsilon^0) = \mathcal{G}(0, \epsilon^0) \mathcal{G}(\epsilon^0, 2e^0) \mathcal{G}(2e^0, \epsilon^0); \\
(c) \quad & \mathcal{G}_n(\epsilon^0) = \mathcal{G}(0, -\epsilon^0) \mathcal{G}(e^0, -\epsilon^0), \quad \mathcal{G}_n(\epsilon^0) = \mathcal{G}(0, -\epsilon^0) \mathcal{G}(e^0, 0) \mathcal{G}(0, 0),
\end{align*}
\]

respectively. They sum up to give the final result \(-\kappa^6 \). We note that U type contributions are cancelled out in the Neumann series of Eq. (B1).

- For \( x = 2\epsilon^0 \), the contribution of \( \mathcal{O}(\kappa^4) \) gives zero, since \( \mathcal{G}(2\epsilon^0) = \mathcal{G}(2\epsilon^0) \) and \( \mathcal{G}(2\epsilon^0) = \mathcal{O}(\kappa^4) \). The contribution of \( \mathcal{O}(\kappa^6) \) gives zero by the come and go property of Eq. (B13). Next, there are two types of contributions of order \( \mathcal{O}(\kappa^6) \) to the series of Eq. (B1). One of them is a U type contribution coming from, for example, \( 0 \rightarrow \epsilon^j \rightarrow \epsilon^j + \epsilon^j \rightarrow 2\epsilon^0 + \epsilon^j \rightarrow 2\epsilon^0 \), their sum in Eq. (B1) giving zero. There are also vertical backtracking contributions which contribute to zero in the Neumann series also. More explicitly, in the Neumann series we have, for example, the \( \mathcal{O}(\kappa^6) \) path: \( 0 \rightarrow \epsilon^j \rightarrow 2\epsilon^0 \rightarrow 3\epsilon^0 \rightarrow 2\epsilon^0 \) with

\[
\Lambda(2\epsilon^0) = \mathcal{G}(2\epsilon^0) + \mathcal{G}(2\epsilon^0) + \mathcal{G}(2\epsilon^0),
\]
with

\[ G_n^2(2e^0) = G_n(0, 3e^0)G_n(3e^0, 2e^0), \]
\[ G_n^2(2e^0) = G_n(0, 2e^0)G_n(2e^0, 3e^0)G_n(3e^0, 2e^0), \]
\[ G_n^2(2e^0) = G_n(0, e^0)G_n(3e^0, 2e^0), \]
\[ G_n^2(2e^0) = G_n(0, e^0)G_n(e^0, 2e^0)G_n(2e^0, 3e^0)G_n(3e^0, 2e^0), \]

and \( G_n^2(2e^0) = G_n^3(2e^0) = G_n^4(2e^0) \). We also need to consider: \( 0 \to e^0 \to 2e^0 \to e^0 \to 2e^0, 0 \to e^0 \to 0 \to e^0 \to 2e^0 \) and \( 0 \to -e^0 \to 0 \to -e^0 \to 2e^0 \). We are left with the contribution of \( \mathcal{O}(\kappa^{10}) \).

- For \( x = e^1 + e^2 \) the first non-zero contribution comes from \( \Lambda(x) = G_n(x) - G_n^2(x) + \mathcal{O}(\kappa^8) \) with \( G_n(x), G_n^2(x) = \mathcal{O}(\kappa^4) \). Note that \( G_n^2(x) = G_n(0, y)G_n(y, x) \) and we must take into account contributions coming from \( y = e^1, e^2 \), next, the use of Eq. (B6) gives the result. The same procedure can be applied to \( x = e^1 + e^3 \).

- For \( x = e^0 + e^1 \), \( G_n^2(x) = G_n(0, y)G_n(y, x) \) for \( y = e^0, e^1 \) and \( G_n(x) = \mathcal{O}(\kappa^4), G_n(0, y), G_n(y, x) = \mathcal{O}(\kappa^2) \). The contribution of \( \mathcal{O}(\kappa^6) \) is zero by imbalance of fermion fields. We are left with \( \mathcal{O}(\kappa^8) \).

- For \( x = e^0 + e^1 + e^2 \) we will show that \( \Lambda(x) = e^0 + e^1 + e^2 = \mathcal{O}(\kappa^8) \). This result improves the bounds coming from hyperplane decoupling method calculations, which, in this case gives \( |\Lambda(x = e^0 + e^1 + e^2)| \leq 4\kappa^2|\kappa|^4(1-1) + 2x^2 = |\kappa|^6 \). Other points, such as \( x = e^0^2 + 2e^1, x = 2e^0 + e^1^2 \) and \( x = 3e^0 \) in the calculating \( \Lambda \) present similar cancellations in the Neumann series and, hence, our calculation improves the global bounds of Theorem 3. The details in those cases are left aside, but can be reproduced following the steps below. The first contribution to \( \Lambda(x = e^0 + e^1 + e^2) \) we need to consider is \( \Lambda(x) = \mathcal{O}(\kappa^6) \) and comes from, recalling Eq. (B1) (in what follows we take \( x = e^0 + e^1 + e^2 \)),

\[ G_n(0, x) = (2c_2 \delta_{\bar{\alpha} \bar{\gamma}} + 2c_2^2)\delta_{\alpha \beta} \kappa^6 \]
\[ G_n(0, y)G_n(y, x) = \begin{cases} 
 2c_2 \delta_{\bar{\alpha} \bar{\gamma}} \delta_{\alpha \beta} \kappa^6 & ; y = e^0, e^1 + e^2 \\
 2c_2^2 \kappa^6 & ; y = e^1, e^2, e^0 + e^1, e^0 + e^2 
\end{cases} \]
\[ G_n(0, y)G_n(y, z)G_n(z, x) = c_2^3 \kappa^6 \ ; y = e^0 ; z = e^0 ; \rho, \sigma \in \{0, 1, 2\}, \rho \neq \sigma, \]

where we made use of the non-intersecting formula (B6) to calculate \( G_n(u, v) \). Note that \( G_n^i = \mathcal{O}(\kappa^8), i \geq 4 \), does not contribute to \( \mathcal{O}(\kappa^6) \). Summing up we get

\[ (\Lambda_{\alpha \beta}(x)) = [-2c_2 \delta_{\bar{\alpha} \bar{\gamma}} + 2c_2^2 \delta_{\alpha \beta} + 2c_2 \delta_{\bar{\alpha} \bar{\gamma}} \delta_{\alpha \beta} + 8c_2^2 \delta_{\alpha \beta} - 6c_2^2 \delta_{\alpha \beta}] \kappa^6 = 0. \]

- For \( x = 3e^0 \), similar to the case \( x = 2e^0 \) we have cancellations in the Neumann series until and including \( \mathcal{O}(\kappa^8) \). For contributions \( \mathcal{O}(\kappa^{10}) \) we have also, U type and vertical paths. U type path is given by, for example, \( 0 \to e^0 \to e^0 + e^1 \to 2e^0 + e^1 \to 2e^0 \to 3e^0 \), etc, summing up to zero in Eq. (B1). Vertical contributions, as for \( x = 2e^0 \), up to and including \( G_n^2 \), sum up to zero in Eq. (B1).


