Baryon-Baryon Bound States From First Principles in $3 + 1$ Lattice QCD With Two Flavors and Strong Coupling

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We determine baryon-baryon bound states in $3 + 1$ dimensional SU(3) lattice QCD with two flavors, $4 \times 4$ spin matrices, and in an imaginary time formulation. For small hopping parameter $\kappa > 0$ and large glueball mass (strong coupling), we show the existence of three-quark isospin $1/2$ particles (proton and neutron) and isospin $3/2$ baryons (delta particles), with asymptotic masses $-3 \ln \kappa$ and isolated dispersion curves. Baryon-baryon bound states of isospin zero are found with binding energy of order $\kappa^2$, using a ladder approximation to a lattice Bethe-Salpeter equation. The dominant baryon-baryon interaction is an energy-independent spatial range-one attractive potential with an $O(\kappa^2)$ strength. There is also attraction arising from gauge field correlations associated with six overlapping bonds, but it is counterbalanced by Pauli repulsion to give a vanishing zero-range potential. The overall range-one potential results from a quark, antiquark exchange with no meson exchange interpretation; the repulsive or attractive nature of the interaction depends on the isospin and spin of the two-baryon state.

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1. INTRODUCTION

One fundamental problem in particle physics is to determine the low-lying energy-momentum (EM) spectrum of Quantum Chromodynamics (QCD). A convenient ultraviolet cutoff version is given by the Wilson lattice QCD model [1–4]. In the strong coupling regime, the infinite volume limit can be reached, hadrons are seen as tightly-bound states of quarks, and confinement is manifested.

In a recent series of papers [5–11], we started a research program aiming at understanding, from first principles, the hadronic particles and their bound states, in the context of imaginary-time formulation of SU(3) lattice QCD with strong coupling (small hopping parameter $0 < \kappa \ll 1$, and large glueball mass). Our goal is to bridge the gap between QCD and nuclear physics, to understand when and how bound states occur and how their binding is related to the effective Yukawa meson-exchange theory. We determined the low-lying energy-momentum (EM) spectrum for increasingly complex SU(3) QCD lattice models, using an imaginary-time formulation. The simplest case in which a two-baryon bound state appears is in the two-flavor total isospin $I = 0, 1$ sectors with $2 \times 2$ spin matrices in $2 + 1$ dimensions; there are no bound states for $I = 2, 3$. However, this model is not complex enough to accommodate protons and neutrons in the one-particle spectrum.

Here, we consider the more realistic two-flavor case in $3 + 1$ dimensions with $4 \times 4$ Dirac spin matrices, which has a global SU(2) isospin symmetry. The following two ingredients are basic in our analysis: i-) derivation of spectral representations for the two- and four-baryon correlation functions, via Feynman-Kac formulas, which allow us to relate complex momentum singularities with the one- and two-baryon EM spectrum; ii-) a lattice version of the Bethe-Salpeter (B-S) equation in a ladder approximation. The spectral representations are new. Other approaches to the particle spectrum, both theoretical or numerical, are given in Refs. [1, 12–24]

It must be pointed out that the detection of particle (and their bound states) masses from the exponential decay rate of suitable correlations, without a spectral representation, is meaningless, and the resulting values may be far from the correct ones, especially in cases where degeneracies are broken with small separations.

Concerning our results, we first show the existence of twenty, three-quark, one-particle states with isolated dispersion curves (upper gap property), and also their associated antiparticles, which includes the proton ($p$), the neutron ($n$) and the delta ($\Delta$) particles. These one-baryon spectral results are exact and, making the lower order explicit, their asymptotic masses are $-3 \ln \kappa - 3\kappa^3/4 + O(\kappa^7)$; if there is mass splitting it is due to contributions of $O(\kappa^7)$ or higher. The upper gap property in the Hamiltonian formulation is unknown.

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Next, we determine the two-baryon bound states in the $I = 0$ sector and below the two-baryon threshold, which is given by twice the smallest of the baryon masses. We find several bound states with binding energies of order $\kappa^2$. The most strongly bound, bound states are given by $\Delta - \Delta$, total spin $S = 3$ states, and also by a superposition of $p - n$ and $\Delta - \Delta$ total spin $S = 0$ states, and also with $\Delta - \Delta$, $S = 2$ bound states. In contrast to the $I = 0$ states treated here, we have found that for the maximum isospin $I = 3$ sector there are bound states in the lowest total spin sectors $S = 0,1$ and no bound states if $S = 2,3$. These results are in agreement with our previous results of Ref. [10] that the attraction between the two particles decreases with increasing $I$. Moreover, as before, there are two sources of attraction, namely, i) the exchange of a quark and an antiquark, which is not a meson particle exchange, and ii) gauge field correlation effects associated with six overlapping bonds. We point out that this work provides the main ingredients for a rigorous treatment of the model, going beyond the ladder approximation (see Refs. [25, 26]).

2. THE MODEL AND THE ONE-BARYON SPECTRUM

We now introduce our SU(3) QCD lattice model and show how our results are obtained. The partition function is given formally by $Z = \int e^{-\mathcal{S}(\psi, \bar{\psi}, g)} \, d\psi \, d\bar{\psi} \, d\mu(g)$, where $S \equiv S(\psi, \bar{\psi}, g)$ is the Wilson action

$$S = \frac{k}{2} \sum \bar{\psi}_{alpha}(u) \Gamma_{alpha}^\sigma \left( g_{u+u+\sigma} \right) \psi_{beta}(u) \left( u + \sigma \hat{e} \right) + \sum_{\mu \in Z_3^d} \bar{\psi}_{alpha}(u) M_{alpha, beta} \psi_{beta}(u) - \frac{1}{g_0^2} \sum_{p} \chi(g_p).$$

Here, besides the sum over repeated indices $\alpha, \beta = 1, 2, 3, 4$ (spin), $a = 1, 2, 3$ (color) and $f = +1/2, -1/2 \equiv +, -$ (isospin), the first sum runs over $u = \left( u_0, u_1, u_2, u_3 \right) \in \mathbb{Z}_d^3 \equiv \left\{ 1/2, 1/2, 3/2, 5/2 \right\} \times \mathbb{Z}_3^3$, $\sigma = \pm 1$ and $\mu = 0, 1, 2, 3$. For $F(\psi, \bar{\psi}, g)$, the normalized expectations are denoted by $\langle F \rangle$. For more details about notation and on the treatment of symmetries such as gauge, SU(2) isospin, charge conjugation, parity, time-reversal and rotational symmetry, we refer to Refs. [6, 7, 10, 11]. Letting $e^\kappa, \mu = 0, 1, 2, 3$, denote the unit lattice vectors, there is a gauge group matrix $U(g_{u+e^\mu \sigma} = U(g_{u+e^\mu} = U(1))^{-1}$ associated with the directed bond $u, u + e^\mu$. We take $0 < g_0^2 < \kappa < 1$. The parameter $m > 0$ is fixed such that $M_{alpha, beta} = M_{beta, alpha} (\delta$ denoting the Kronecker delta) and $M \equiv M(\kappa) = m + 2 \kappa = 1$.

The choice of the shifted lattice for the time direction, avoiding the zero-time coordinate, is so that, in the continuum limit, two-sided equal time correlations of quark Fermi fields correlations can be accommodated.

The action $S$ is one of a family of actions which has no spectral doubling for the free fermions, and the free fermion dispersion curve is increasing in each momentum component, convex for small momenta. The SU(2) spin symmetry, we refer to Refs. [6, 7, 10, 11]. Letting $e^\kappa, \mu = 0, 1, 2, 3$, denote the unit lattice vectors, there is a gauge group matrix $U(g_{u+e^\mu \sigma} = U(g_{u+e^\mu} = U(1))^{-1}$ associated with the directed bond $u, u + e^\mu$. We take $0 < g_0^2 < \kappa < 1$. The parameter $m > 0$ is fixed such that $M_{alpha, beta} = M_{beta, alpha} (\delta$ denoting the Kronecker delta) and $M \equiv M(\kappa) = m + 2 \kappa = 1$.

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The fields obey the normalization $\langle B_i \bar{B}_j \rangle^{(0)} = -\delta_{ij}$, where $i,j$ are collective indices and the superscript $(0)$ denotes $\kappa = 0$ in the hopping terms in the action. In general, below, the superscript $(n)$ means the coefficient of $\kappa^n$, $n=0,1,\ldots$

In analogy with the two-flavor SU(3) gauge QCD, with only up ($f = +1/2$) and down ($f = -1/2$) quarks, we identify the particles associated with the fields in Eq. (1) as proton, neutron, $\Delta^{++}$, $\Delta^-$, $\Delta^*$, $\Delta^0$ and $\Delta^0$.

We treat the one-baryon states adapting the methods of Refs. [6, 10]. The baryon-baryon correlation function is defined by ($\chi$ here denotes the characteristic function and $^*$ complex conjugation)

$$G_{t_1t_2}(u, v) = \langle B_{t_1}(u) \bar{B}_{t_2}(v) \rangle \chi_{u^0 \leq v^0} - \langle \bar{B}_{t_1}(u) B_{t_2}(v) \rangle^* \chi_{u^0 \geq v^0} \equiv G_{t_1t_2}(u - v),$$

where $t = (I, I_z, s)$ is a collective index. It is important to stress that the apparently awkward form of this correlation emerges naturally from the two-time orderings in the Feynman-Kac formula. By isospin symmetry, $G_{t_1t_2}(u = v)$ is diagonal in $I, I_z$ and, for $I$ fixed, independent of $I_z$. Also, the $z$-component of spin $s$ takes values $\pm 1/2$ for $I = \pm 1$ or $\pm 3/2, \pm 1/2$ for $I = 3/2$. Setting $x^0 \equiv u^0 - v^0 \neq 0$ and dropping the isospin indices, from the Feynman-Kac formula and taking the Fourier transform $\tilde{G}_{t_1t_2}(p) = \sum_{x \in \mathbb{Z}^3} G_{t_1t_2}(x) e^{-ipx}$ of $G_{t_1t_2}(x)$, we obtain the following spectral representation for $\tilde{G}$

$$\tilde{G}_{s_1s_2}(p) = \tilde{G}_{s_1s_2}(\bar{p}) - (2\pi)^3 \int_1^1 f(p^0, \lambda^0) d\lambda \alpha_{\bar{p},s_1s_2}(\lambda^0), \quad (2)$$

where $d\alpha \alpha_{\bar{p},s_1s_2}(\lambda^0) = \int_{\mathbb{T}^3} \delta(\bar{p} - \tilde{\lambda}) d\lambda d\delta(\bar{B}_{s_1}(1/2, \tilde{\lambda})) \cdot \delta(\bar{B}_{s_2}(1/2, \tilde{\lambda})) \cdot \delta I \cdot \delta (\bar{T}, \bar{S})$, $\mathbb{T} = (-\pi, \pi]$ is the product of the spectral families for the operators $\bar{T}_0, \bar{T}_1, \bar{T}_2$ and $\bar{T}_3$, $f(x, y) \equiv e^{ixx} - e^{-ixx} - 1 + e^{-ixx} - y^{-1}$. And we set $\tilde{G}(p) = \sum_{\lambda} e^{-ipx} G(x^0 = 0, \tilde{\lambda})$. Using symmetries, $\tilde{G}(\tilde{p})$ is diagonal and independent of the spin, at least up to $\mathcal{O}(\kappa^\delta)$.

The spectral representation given in Eq. (2) allows us to identify complex $p$ singularities of $\tilde{G}(p)$ with points in the EM spectrum. Indeed, the twenty one-baryon isolated dispersion curves are all equal up to and including $\mathcal{O}(\kappa^5)$ and, with $p_T^2 \equiv \sum_{i=1}^3 (1 - cos p^i)$, are given by

$$w(p) = [-3 \ln \kappa - 3\kappa^3/4 + p_T^2 \kappa^3/8] + \mathcal{O}(\kappa^7),$$

so that $w(p)$ is convex for small $|p_T|$. Actually, we know the $w(p) = 0$ are equal up to $\mathcal{O}(\kappa^7)$. The spectral measure $d\lambda \alpha_{\bar{p},s_1s_2}(\lambda^0)$ has the approximate decomposition $d\lambda \alpha_{\bar{p},s_1s_2}(\lambda^0) = Z_{s_1s_2}(\tilde{p})\delta(\lambda^0 - e^{-w(\tilde{p})}) d\lambda^0 + dv_{s_1s_2}(\lambda^0, \tilde{p})$, where the first term corresponds to the one-particle contribution. Letting $\tilde{\Gamma}_{s_1s_2}(p)$ denote the analytic extension of $\tilde{G}_{s_1s_2}(p)^{-1}$ up to near the two-baryon threshold, we have $Z_{s_1s_2}(\tilde{p})^{-1} = -(2\pi)^3 e^{w(\tilde{p})} \frac{\partial^2 w}{\partial \lambda^0}(\tilde{p}^0 = i\chi, \tilde{p})\chi = w(\tilde{p})$, and the $\lambda^0$ support of $dv_{s_1s_2}(\lambda^0, \tilde{\lambda})$ is contained in $|\lambda^0| \leq |\kappa|^\epsilon$.

### 3. BARYON-BARYON BOUND STATES

Here we consider the subspace of the physical Hilbert space consisting of two-baryon states generated by the product of one-baryon fields, $\bar{B}_{t_1}(x_1) \bar{B}_{t_2}(x_2)$, and determine baryon-baryon bound states below the two-baryon threshold. The method employs a B-S equation for a suitable set of four-point functions.

If we ignore possible linear dependencies, the above subspace has dimension $400 = 20 \times 20$, which decomposes into two-baryon total isospin $I = 0, 1, 2, 3$ sectors, of dimensions 20, 108, 160, 112, respectively. Note that, because of Pauli exclusion, the dimension of this subspace is reduced if $x_1 = x_2$. Hence, any analysis that only considers coincident points does not give information on the whole two-baryon subspace.

To reduce the algebraic complexity of the analysis, hereafter we restrict our attention to the $I = 0$ sector, where we expect to find a deuteron, $p - n$ bound state. The $I = 0$ states are given by the isospin Clebsch-Gordan (C-G) linear combinations

$$\phi_{s_1s_2}^{1/2}(x_1, x_2) = \sum_{t_1, t_2 = 0}^{1/2} c_{t_1t_2} \bar{B}_{t_1}(x_1) \bar{B}_{t_2}(x_2),$$

for the coupling of two 1/2 isospin baryons, and similarly, with $(1/2 \rightarrow 3/2, s_1 \rightarrow t_1)$, for the coupling of two 3/2 isospins. We use $s_j (t_j)$ for isospin 1/2 (3/2) baryons and recall that the 1/2, 3/2 coupling does not give $I = 0$. We refer to this description of the two-particle states as the individual spin basis. It turns out that the dominant interaction between baryons admits a simpler description in terms of total spin. The total spin states are obtained by taking C-G combinations of the $\tau$'s. Namely, $\tau^{1/2}_{SSZ}(x_1, x_2) = \sum_{s_1 + s_2 = s} a_{s_1s_2}^s \phi_{s_1s_2}^{1/2}(x_1, x_2)$, and similarly, with $(1/2 \rightarrow 3/2, S \rightarrow T, s_1 \rightarrow t_1)$, where the $a$'s are the spin C-G coefficients. Using their properties, we find that $\tau^{1/2}_{SSZ}(x_1, x_2) = (-1)^{l+1} \tau^{1/2}_{TSS}(x_2, x_1)$ and $\tau^{3/2}_{TTZ}(x_1, x_2) = (-1)^{l+1} \tau^{3/2}_{TTZ}(x_2, x_1)$, so that for coincident points the
states vanish for \( S = 0 \), and for \( T = 0, 2 \), a kinematic statistical effect; they are space symmetric (antisymmetric) for \( S = 1, T = 1, 3 \) (\( S = 0, T + 0, 2 \)).

We order the individual spin pairs as \((s_1, s_2) = (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})\); \((t_1, t_2) = (\frac{3}{2}, \frac{3}{2}), (\frac{3}{2}, -\frac{1}{2}), (\frac{1}{2}, 1), (-\frac{1}{2}, -\frac{1}{2}), \ldots, (-\frac{1}{2}, -\frac{1}{2}), \ldots, (-\frac{1}{2}, -\frac{1}{2})\), and the total spin pairs as \((S, S_z) = (1, 1), (T, T_z) = (1, 1), (S, S_z) = (1, 0), (T, T_z) = (1, 0), (S, S_z) = (3, 1), (2, 1), \ldots, (2, -2)\). The individual and the total spin basis are related by a real orthogonal transformation whose entries are C-G coefficients.

As we do not know a priori which linear combination of states \( S \) corresponds to a bound state, we consider the \( 20 \times 20 \) matrix of (truncated) four-baryon functions

\[
M(x_1, x_2; x_3, x_4) = \left( \begin{array}{cc}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{array} \right)(x_1, x_2; x_3, x_4),
\]

where, in the individual spin basis, suppressing the lattice coordinates \((x_1, x_2; x_3, x_4)\), and for \( x_0^1 = x_0^2 = x_0^3 = x_0^4 \), \( x_1^0 \leq x_0^3 \), \( M_{11} \), \( M_{12} \), \( M_{21} \), \( M_{22} \) is well defined. As in Ref. [10], we modify the B-S equation by interchanging the barred and unbarred fields and taking the complex conjugate. As in Ref. [10], \( x_1, \ldots, x_4 \) are now taken in \( \mathbb{Z}^2 \). \( M_{ij} \) has a spectral representation which we use to detect bound states below the two-baryon threshold.

Next, we write a B-S equation for \( M \) as,

\[
M = M_0 + M_0 K M.
\]

Formally, this equation defines the B-S operator \( K = (M_0)^{-1} - (M)^{-1} \). Our strategy is to compute \( K \) in the leading \( \kappa \) order, which we call a ladder approximation to \( L \), and then solve the B-S equation for \( M \) using \( L \).

After taking the Fourier transform in time, the B-S equation is similar to a two-particle Schrödinger operator resonant equation with an interaction potential determined by \( K \). More precisely, in lattice QCD, \( K \) has the interpretation of an energy-dependent nonlocal potential (see Refs. [10, 11]).

In the B-S equation (3), \( M_0 \) is obtained from \( M \) by erroneously applying Wick’s theorem to the one-baryon composite fields \( \hat{B} \) in \( M \), and reduces to the product of two exact two-baryon correlations. In the total spin basis, \( M_0 \) and \( M_0 \) are matrix operators on \( \ell'_s = \ell'_s \oplus \ell'_s \), such that \( f_1 + f_2 = \ell'_s \); \( f_1 \) (\( f_2 \)) is associated with the first 13 (last 7) components: \( f_1 \in \ell'_s(A_1), f_2 \in \ell'_s(A_2) \), where \( s(a) \) means the symmetric (antisymmetric) subspace and \( A_s = \{(x_1, x_2; j) \in \mathbb{Z}^4 \times \mathbb{Z}^4 \times (1, 2, \ldots, 13)|x_1^0 = x_2^0\} \) and \( A_a = \{(x_1, x_2; j) \in \mathbb{Z}^4 \times \mathbb{Z}^4 \times (14, \ldots, 20)|x_1^0 = x_2^0; x_1 \neq x_2\} \).

Using the orthogonality properties of the two-point functions, in the total spin basis, we have

\[
M_{0, S, S', S''}(x_1, x_2; x_3, x_4) = \delta_{S S'} \delta_{S''} \delta(x_1 - x_3) \delta(x_2 - x_4) + L \delta(x_1 - x_4) \delta(x_2 - x_3),
\]

and the same for \( M_{0, T, T', T''} \). Hence, \( M_0^{(0)} \) acts as \(-2 \mathbb{1} \neq 0 \) on \( \ell'_s \), and \( M_0 \) is invertible. To define the B-S operator \( K = M_0^{-1} - M^{-1} \), we need \( M_0^{-1} \). Due to the linear dependence relation at coincident points \( T = 0 \), \( M_0^{-1} \) has a null space in the symmetric subspace of \( \ell'_s \). For this reason, we are forced to reduce the symmetric subspace for coincident points. For coincident points (denoted by zero argument), in the total spin basis, by a lengthy computation we find \( M_0^{(0)}(0) = M_1^{(0)}(0) = M_2^{(0)}(0) = M_3^{(0)}(0) \).

where \( M_1^{(0)}(0) = -1 \mathbb{1} \) on the components from 7 to 13. The eigenvalues of \( M_1^{(0)}(0) \) are \(-4 \mathbb{1} \), and we take the restriction of \( M_0^{(0)}(0) \) to the orthogonal complement of the null space in the symmetric subspace. So, for coincident points we have a 10 dimensional space. For non-coincident points, \( M_0^{(0)}(0) \) agrees with \( M_0^{(0)}(0) \neq 0 \), so that \( K \) is well defined. As in Ref. [10], we modify \( M_0 \) at coincident points by a multiplicative constant \( 1/2 \). Precisely, we modify \( M_0 \) by replacing \( M(x_1, x_2; x_3, x_4) \) by \( M'(x_1, x_2; x_3, x_4) = h(x_1, x_2)M(x_1, x_2; x_3, x_4)h(x_3, x_4) \), where \( h(x,y) = |1 - \delta(x - y)| + \frac{1}{2}\delta(x - y) \) and the B-S equation becomes \( M' = M_0 - M_0 K M' \). For simplicity of notation, we drop the prime and keep using \( M \) and \( K \). By doing this, we improve the temporal decay properties of \( K \) and avoid having to deal with energy-dependent potentials (see Ref. [11]). \( K \) admits a Neumann expansion representation.

\[
M = M_0 - M_0 \delta M_0 = M_0 - M_0 \delta M_0, \quad \delta M_0 = (M_0^{-1})^{-1}(0) - (M_0)^{-1}(0),
\]

and, since \( \delta M_0 \) is \( \mathcal{O}(\kappa^2) \) and \( \delta M_0 \) is \( \mathcal{O}(\kappa^3) \), the ladder approximation to \( K \) is \( L \equiv K^{(2)}(2) = (M_0)^{-1}(\delta M_0)(M_0)^{-1}(0) \). Its kernel is given by \( K^{(2)}(2)(x_1, x_2; x_3, x_4) = \)
\[ \frac{1}{\Xi(2)} \sum_{\sigma, \gamma} \delta(x_2 - x_1 - \sigma \epsilon^1) \delta(x_2^0 - x_1^0) \delta(x_1 - x_3) \delta(x_2 - x_4) \] where, in the individual spin basis, \( \psi^{(2)} \) is

\[ \psi^{(2)}_{s_1 s_2 s_3 s_4} = \sum_{\ell_1 \ell_2 \ell_3 \ell_4} C_{\ell_1 \ell_2 \ell_3 \ell_4} \psi_{1 \ell_1, \ell_2, \ell_3, \ell_4} \psi_{a_1, a_2, a_3, a_4} \] 

with all fields at the same site, and where the prime means the sum over \( I_1, ..., I_4 \), with \( I_1 + I_2 = 0 = I_3 + I_4 \). Here, we have made use of the translation and rotational symmetries. A similar expression holds for the components \( t_1 t_2 s_3 s_4, s_1 s_2 t_3 t_4 \) and \( t_1 t_2 s_3 t_4, t_1 t_2 l_3 l_4 \). By inspection, since the \( B \) fields have lower spin indices, the \( \psi \) and \( \tilde{\psi} \) in the above means must have both spin indices either lower or upper. This is a quark-antiquark exchange, but since a meson particle has one upper and one lower spin index (see Refs. [7, 11]), this is not a meson particle exchange, and we refer to it as a quasi-meson exchange. By a lengthy computation, it turns out that \( \psi^{(2)} \) has an especially simple, almost diagonal form in the total spin basis given by \( \psi^{(2)} = W + W + W \oplus W_{1d} \oplus W_2 \oplus W_{2d} \), with \( W = \frac{1}{3} \left( \begin{array}{cc} -1 & -\sqrt{80} \\ -\sqrt{80} & 1 \end{array} \right) \) (with eigenvalues \( \pm 3 \)), \( W_2 = \left( \begin{array}{ccc} 7 & 4 & 1 \\ 4 & 1 & 4 \end{array} \right) \) (with eigenvalues \( -1 \) and \( 9 \)), \( W_{1d} = -3P_{13}, W_{2d} = -P_{22} \), where \( P_{13} \) is the orthogonal projection on the 7th through 13th components and \( P_{22} \) on the 16th through 20th.

We return to the determination of bound states and solve the B-S equation using \( L \) (see Refs. [10, 11] for details). In \( M \) and \( M_0 \), we first pass to the lattice relative coordinates \( \xi = x_2 - x_1, \eta = x_4 - x_3 \), and \( \tau = x_3 - x_2 \), take the Fourier transform in the variable \( \tau \), with dual variable \( (k^0, \vec{k}) \), and set the system momentum \( \vec{k} = 0 \). The new quantities are denoted with a hat and have arguments \( (\vec{\xi}, \vec{\eta}, k^0) \). (Note that \( M \) and \( M_0 \) have the same \( k^0 \) singularities!) In our ladder approximation \( L = \kappa^2 \psi^{(2)} \sum_{\sigma, \gamma} \delta(\vec{\tau} - \vec{\xi}) \delta(\sigma \vec{\gamma} - \vec{\zeta}) \), and the B-S equation has the solution

\[ M(\vec{\xi}, \vec{\eta}, k^0) = M_0(\vec{\xi}, \vec{\eta}) + \sum_{\sigma_1, \sigma_2, j_1, j_2} M_0(\vec{\xi}, \vec{\sigma}_1, k^0) \kappa^2 \psi^{(2)}(1 - \kappa^2 \psi^{(2)} M_0)_{\sigma_1, \sigma_2, j_1, j_2}^{-1} \langle \sigma_1 \vec{e}_1, \sigma_2 \vec{e}_2, \eta \rangle M_0(\sigma_2 \vec{e}_2, \eta), \]

where the \( M_0 \) lattice indices are restricted to the spatial nearest neighbors of zero, and we suppress the \( k^0 \) dependence and spin indices on the r.h.s. Reinstating the spin indices, the resulting matrix is \( 120 \times 120 \).

We determine bound states below the two-baryon threshold as singularities in \( k^0 = i(2m - \epsilon) \), where \( m \) denotes the minimum of the baryon masses and \( \epsilon > 0 \) is the bound state binding energy. To go further, we need properties of \( M_0 \). Starting from the spectral representation for the two-baryon function, and making a spin diagonal approximation to \( G \), the dominant contribution to \( M_0 \) is given by, in the total spin basis,

\[ M_{0, SS, S', S'}(\vec{\xi}, \vec{\eta}, k^0) = \sum_{\lambda_0} e^{i \vec{p} \cdot \vec{\xi}} \sum_{\lambda_0} e^{i \vec{p} \cdot \vec{\eta}} \sum_{\lambda_0} e^{i \vec{p} \cdot \vec{\eta}} \sum_{\lambda_0} e^{i \vec{p} \cdot \vec{\eta}} [\Xi(\vec{p}, \vec{\xi}, \vec{\eta}) \delta(\vec{p} \cdot \vec{\xi} \oplus \vec{p} \cdot \vec{\eta} \oplus \vec{p} \cdot \vec{\eta})] \delta_{S, S} \delta_{S', S'}, \]

where \( \Xi(\vec{p}, \vec{\xi}, \vec{\eta}) = \delta_{S_1} \cos \vec{p} \cdot \vec{\xi} \cos \vec{p} \cdot \vec{\eta} + \delta_{S_0} \sin \vec{p} \cdot \vec{\xi} \sin \vec{p} \cdot \vec{\eta} \), and the same for \( T = 1, 3 \) (\( T = 0 \) replacing \( S = 1 \) \( S = 0 \). We further make the following approximations to \( M_0 \): a) retain only the product of one-particle contributions; b) \( u(\vec{p}) = m \approx k^0 \vec{p}/8; c) m \approx -3 \ln \kappa \) and d) \( Z(\vec{p}) \approx -2(\pi)^{-3} e^{-u(\vec{p})} \approx -2(\pi)^{-3} k^3 \). With these approximations, the singularities of \( M \), below the two-baryon threshold, occur as zeroes of the determinant of the \( 120 \times 120 \) matrix \[ 1 - \kappa^2 \psi^{(2)} M_0(\sigma_1 e^1, \sigma_2 e^2, \eta) \]. Anticipating that the binding energy is of order \( \kappa^2 \), we also approximate the denominator of \( M_0 \) in \( M_0 = \sum_{\lambda_0} e^{i \vec{p} \cdot \vec{\xi}} \sum_{\lambda_0} e^{i \vec{p} \cdot \vec{\eta}} \sum_{\lambda_0} e^{i \vec{p} \cdot \vec{\eta}} [\Xi(\vec{p}, \vec{\xi}, \vec{\eta}) \delta(\vec{p} \cdot \vec{\xi} \oplus \vec{p} \cdot \vec{\eta} \oplus \vec{p} \cdot \vec{\eta})] \delta_{S, S} \delta_{S', S'}. \]

We refer to the spectral decomposition of \( \psi^{(2)} \) above, and recalling the ordering of the total spin basis, we see that the most strongly bound, bound states (\( \lambda = -3 \)) are associated with a superposition of \( p - n \) and \( \Delta - \Delta \) total spin 1 states; and also with \( \Delta - \Delta \) total spin 3 states. The more weakly bound, bound states (\( \lambda = 1 \)) are associated with a superposition of \( p - n \) and \( \Delta - \Delta \) total spin 0 states; and also with \( \Delta - \Delta \) total spin 2 states.

Similarly, applying our method to the maximum \( I = 3 \) sector (freezing all isospins to +), we find no bound states if \( S = 1, 2 \), but bound states do appear if \( S = 0, 1 \). Moreover, the binding for the \( S = 0 \) case is stronger than the one for \( S = 1 \). This is similar to the bound state results of Ref. [10] regarding 2 + 1 dimensions, 2 x 2 spin matrices and two flavors, i.e. with intertwined roles of spin and isospin. We are now working on the more complex \( I = 1, 2 \) sectors.
4. FINAL REMARKS

We note that although our bound state results are obtained using quite complicate machinery, in the end a simple picture emerges for the formation of a baryon-baryon bound state. The two-baryon dynamics in relative coordinates behaves approximately like that of a non-relativistic one-particle lattice Hamiltonian $T + V$ with lattice kinetic energy $-\kappa^2 \Delta / 8$, where $\Delta$ is the spatial lattice Laplacian and the potential energy $V$ is $\kappa^2 V'$, the quasi-meson exchange space range-one potential which dominates the kinetic energy for small $\kappa$. The attractive or repulsive nature of the interaction depends on the isospin, spin spectral structure of $V$ at a single site of space-range one. Because of the $\kappa^3$ dependence of the kinetic energy $T$ and the $\kappa^2$ dependence of the potential energy, there is no minimal critical value of the interaction strength needed for the presence of a bound state.

To conclude, if we consider contributions to the B-S kernel comprised of linear chains of quark, anti-quark pairs they result in an exponentially decreasing potential with decay rate $-2 \ln \kappa$, as for the Yukawa theory. It would be interesting to look for the expected distance$^{-1}$ Ornstein-Zernicke like correction to this potential and to determine the spin and isospin dependence of the binding energy, and the effect of the number of flavors. Especially, the analysis of the SU(3) flavor case can shed some light in understanding bound states when strangeness and, consequently, $\Lambda$ particles are present. Also, there is the problem of determining bound states of more than two baryons. However, lattice effects are expected to be relevant and unrealistic in determining the resulting geometric spatial configuration of possible bound states. Finally, and more importantly, we would like to know how the baryon spectrum and the bound state binding energies behave near the scaling limit.

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